United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Organised by the United Kingdom Mathematics Trust

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Solutions and investigations

November 2021

These solutions augment the shorter solutions also available online. The shorter solutions sometimes leave out details. The solutions given here are full solutions, as explained below. In some cases alternative solutions are given. There are also many additional problems for further investigation. We welcome comments on these solutions and the additional problems. Please send them to enquiry@ukmt.org.uk.

The Senior Mathematical Challenge (SMC) is a multiple-choice paper. Sometimes you can find the correct answers by working backwards from the given alternatives, or by showing that four of them are not correct. This can be a sensible thing to do in the context of the SMC.

However, this does not provide a full mathematical explanation that would be acceptable if you were just given the questions without any alternative answers. So we aim at including for each question a complete solution with each step explained (or, occasionally, left as an exercise), and not based on the assumption that one of the given alternatives is correct. We hope that these solutions provide a model for the type of written solution that is expected when presenting a complete solution to a mathematical problem (for example, in the British Mathematical Olympiad, the Mathematical Olympiad for Girls and similar competitions).

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1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
C	C	C	D	E	B	B	B	D	D	E	A	D	D	C	C	B	C	D	A	A	E	A	E	B

1. Cicely had her 21$^{\text{st}}$ birthday in 1939.

When did she have her 100$^{\text{th}}$ birthday?

A 2020 B 2019 C 2018 D 2010 E 2008

Solution **C**

Cicely had her 21st birthday in 1939. Since 1939 – 21 = 1918, it follows that she was born in 1918.

Now, 1918 + 100 = 2018.

Therefore Cicely's 100th birthday was in 2018.

For investigation

1.1 The mathematician Augustus De Morgan was born and died in the 19th century. On one birthday he noticed that the square of his age was the same as the year number.

In which year was Augustus De Morgan born?

1.2 Determine for which values of n a person born in year n could have the same experience as Augustus De Morgan if they lived long enough, that is, they would have a birthday on which the square of their age was the same as the year number.

2. The sequence, formed from the sequence of primes by rounding each to the nearest ten, begins 0, 0, 10, 10, 10, 10, 20, 20, 20, 30, ...

When continued, how many terms in this sequence are equal to 40?

A 1 B 2 C 3 D 4 E 5

Solution **C**

The integers that round to 40 are those in the range from 35 to 44.

The primes in this range are 37, 41 and 43.

Therefore there are 3 primes that round to 40.

For investigation

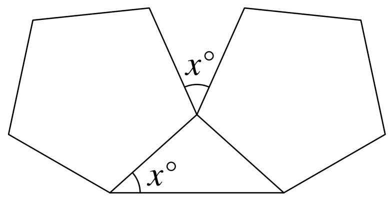
2.1 How many primes are rounded to 50?

2.2 What is the largest number of primes that round to the same multiple of 10?

3. The diagram shows two congruent regular pentagons and a triangle. The angles marked $x°$ are equal.

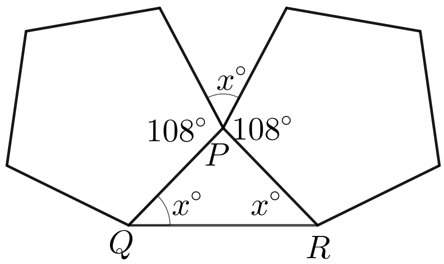
What is the value of x?

A 24 B 30 C 36 D 40 E 45



Solution **C**

Let P, Q and R be the points shown in the diagram.



The interior angles of a regular pentagon are all 108°.

Because the two pentagons are congruent, $PR = PQ$. Therefore $\angle PRQ = \angle PQR = x°$.

Because the sum of the angles in a triangle is 180°, from the triangle PQR, we have $x° + x° + \angle QPR = 180°$. Therefore $\angle QPR = 180° - 2x°$.

The sum of the angles at a point is 360°. Therefore from the angles at the point P, we have

$$(180° - 2x°) + 108° + x° + 108° = 360°.$$

That is

$$396° - x° = 360°.$$

It follows that

$$x° = 396° - 360° = 36°.$$

For investigation

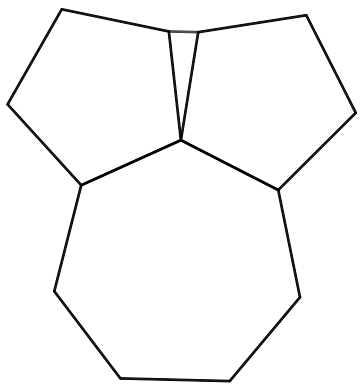
3.1 Prove that the sum of the angles in a triangle is 180°.

3.2 Prove that each interior angle of a regular pentagon is 108°.

3.3 Find a formula in terms of n for the size of the interior angles of a regular polygon with n sides.

3.4 The diagram shows a regular heptagon, two regular pentagons and a triangle.

What are the interior angles of the triangle?



4. The positive integer k is a solution of the equation $(k \div 12) \div (15 \div k) = 20$.

What is the sum of the digits of k?

A 15 B 12 C 9 D 6 E 3

SOLUTION **D**

We have

$$\begin{aligned}(k \div 12) \div (15 \div k) &= \frac{k}{12} \div \frac{15}{k} \\ &= \frac{k}{12} \times \frac{k}{15} \\ &= \frac{k \times k}{12 \times 15} \\ &= \frac{k^2}{180}.\end{aligned}$$

It follows that

$$\begin{aligned}(k \div 12) \div (15 \div k) = 20 &\Leftrightarrow \frac{k^2}{180} = 20 \\ &\Leftrightarrow k^2 = 3600 \\ &\Leftrightarrow k = 60, \text{ as } k > 0.\end{aligned}$$

The sum of the digits of 60 is $6 + 0 = 6$.

FOR INVESTIGATION

4.1 Find the solutions of the following equations.

(a) $(x \div 5) \div (5 \div x) = 4$.

(b) $(x \div 2) \div ((x \div 10) \div (x \div 3)) = 15$.

5. The sum of four consecutive primes is itself prime.

What is the largest of the four primes?

A 37 B 29 C 19 D 13 E 7

SOLUTION **E**

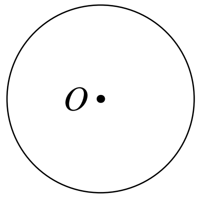
The sum of four odd primes is an even number greater than 2, and therefore is not a prime. Therefore the four consecutive primes whose sum is prime includes the only even prime 2.

It follows that the four consecutive primes are 2, 3, 5 and 7. Their sum is 17 which is a prime. The largest of these four consecutive primes is 7.

FOR INVESTIGATION

5.1 Which is the smallest prime that is the sum of five consecutive primes?

6. Three points, P, Q and R are placed on the circumference of a circle with centre O. The arc lengths PQ, QR and RP are in the ratio $1 : 2 : 3$.

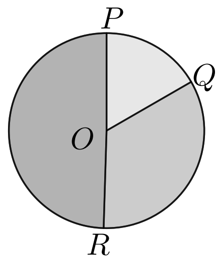


In what ratio are the areas of the sectors POQ, QOR and ROP?

A $1 : 1 : 1$ B $1 : 2 : 3$ C $1 : \pi : \pi^2$
D $1 : 4 : 9$ E $1 : 8 : 27$

Solution **B**

The key facts that we need to use here are



(a) the ratio of the length of an arc to the length of the circumference is the same as the ratio of the angle that the arc subtends at the centre of the circle to the angle in a complete revolution,

and

(b) the ratio of the area of a sector to the area of the circle is the same as the ratio of the angle in the sector to the angle in a complete revolution.

It follows from (a) that the ratios of the arc lengths of PQ, QR and RP are the same as the ratios of the angles that the arcs subtend at the centre of the circle.

Therefore these angles are in the ratio $1 : 2 : 3$.

Similarly, it follows from (b) that the ratios of the areas of the sectors POQ, QOR and ROP are the same as the ratios of the angles in the sectors.

Therefore the areas of the sectors POQ, QOR and ROP are in the ratio $1 : 2 : 3$.

Commentary

From the basic facts (a) and (b) we can deduce that

(c) The ratio of the length of an arc of a circle to the circumference of a circle is equal to the ratio of the area the arc subtends at the centre of the circle to the area of the circle.

An alternative method would have been to base the solution on this fact.

For investigation

6.1 Suppose that the circle has radius 3 and the arc lengths PQ, QR and RP are in the ratio $2 : 3 : 4$.

What is the area of the sector QOR?

7. Which of these numbers is the largest?

A 2^{5000} B 3^{4000} C 4^{3000} D 5^{2000} E 6^{1000}

Solution **B**

We have

$$2^{5000} = (2^5)^{1000} = 32^{1000},$$
$$3^{4000} = (3^4)^{1000} = 81^{1000},$$
$$4^{3000} = (4^3)^{1000} = 64^{1000},$$

and

$$5^{2000} = (5^2)^{1000} = 25^{1000}.$$

Since $6 < 25 < 32 < 64 < 81$, it follows that $6^{1000} < 25^{1000} < 32^{1000} < 64^{1000} < 81^{1000}$.

Therefore $6^{1000} < 5^{2000} < 2^{5000} < 4^{3000} < 3^{4000}$.

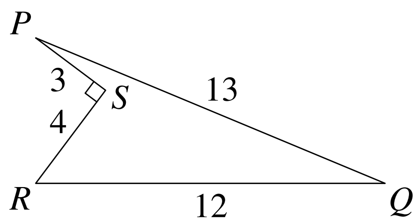
Hence, of the given numbers, it is 3^{4000} that is the largest.

For investigation

7.1 Which of these numbers is the largest?
(a) 2^{7000}, (b) 3^{6000}, (c) 4^{5000}, (d) 5^{4000}, (e) 6^{3000}.

8. What is the area of the region inside the quadrilateral $PQRS$?

A 18 B 24 C 36 D 48
E more information needed

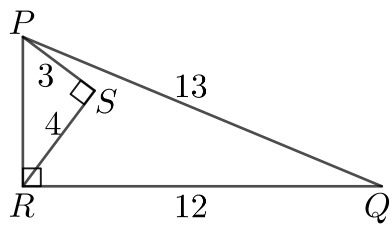


Solution **B**

By Pythagoras' Theorem applied to the right-angled triangle PSR, we have $PR^2 = 3^2 + 4^2 = 9 + 16 = 25$. Therefore $PR = 5$.

It follows that in the triangle PRQ we have

$PQ^2 = 13^2 = 169 = 25+144 = 5^2+12^2 = PR^2+RQ^2$.



Therefore, by the converse of Pythagoras' Theorem, $\angle PRQ = 90°$.

Because $\angle PRQ = 90°$, the area of the triangle PRQ is $\frac{1}{2}(RQ \times RP) = \frac{1}{2}(12 \times 5) = 30$. Similarly, the area of the triangle PSR is $\frac{1}{2}(SP \times SR) = \frac{1}{2}(3 \times 4) = 6$.

The area of the quadrilateral $PQRS$ is the area of the triangle PRQ minus the area of the triangle PSR. Hence the area of $PQRS$ is $30 - 6 = 24$.

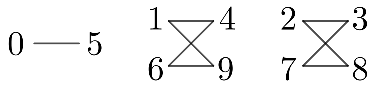
9. Alison has a set of ten fridge magnets showing the integers from 0 to 9 inclusive.

In how many different ways can she split the set into five pairs so that the sum of each pair is a multiple of 5?

A 1 B 2 C 3 D 4 E 5

SOLUTION **D**

The number 0 can only be paired with 5.



The number 1 may be paired with 4 or with 9. If 1 is paired with 4, 6 has to be paired with 9. If 1 is paired with 9, 6 has to be paired with 4.

The number 2 may be paired with 3 or with 8. If 2 is paired with 3, 7 has to be paired with 8. If 2 is paired with 8, 7 has to be paired with 3.

These possibilities are shown in the diagram above. Thus the complete pairing is determined by first the choice which of 4 or 9 to pair with 1, giving two choices, and then the choice of which of 3 or 8 to pair with 2.

These choices are independent. It follows that there are $2 \times 2 = 4$ ways to split the set of numbers into five pairs so that the sum of each pair is a multiple of 5.

FOR INVESTIGATION

9.1 List the 4 different pairings that satisfy the condition that the sum of each pair is a multiple of 5.

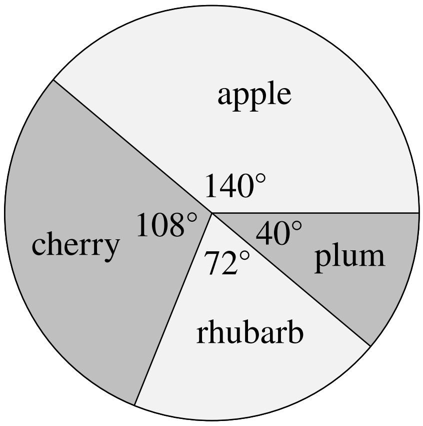
9.2 Bibi has a set of twenty fridge magnets showing the integers from 0 to 19, inclusive. In how many different ways can she split the set into ten pairs so that the sum of each pair is a multiple of 5?

9.3 Chandra has a set of twenty-four fridge magnets showing the integers from 0 to 23, inclusive. In how many different ways can she split the set into twelve pairs so that the sum of each pair is a multiple of 5?

10. In a survey, people were asked to name their favourite fruit pie. The pie chart shows the outcome. The angles shown are exact with no rounding.

What is the smallest number of people who could have been surveyed?

A 45 B 60 C 80 D 90 E 180



Solution **D**

Suppose that p people were surveyed.

The total of the angles is 360°. Therefore the proportion of the people who said that their favourite is apple pie is $\frac{140}{360} = \frac{7}{18}$. Hence the number who chose apple pie was $\frac{7}{18}p$. This is an integer. Therefore p is a multiple of 18.

Similarly, as the proportion who said their favourite is cherry pie is $\frac{108}{360} = \frac{3}{10}$, we deduce that p is a multiple of 10.

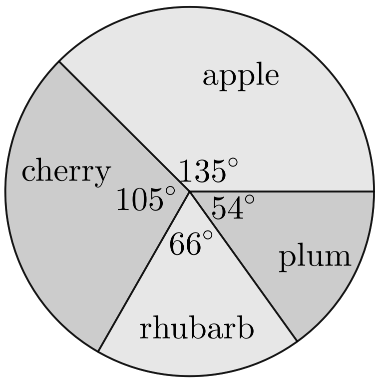
Likewise, as $\frac{72}{360} = \frac{1}{5}$ and $\frac{40}{360} = \frac{1}{9}$, we know that p is also a multiple of 5 and of 9.

Therefore the smallest possible value of p is the least common multiple of 18, 10, 5 and 9, which is 90. Hence the smallest number of people who could have been surveyed is 90.

For investigation

10.1 The results of another survey about people's favourite fruit pies are shown in the pie chart on the right. Again, the angles are exact with no rounding.

What is the smallest number of people who could have been surveyed?



11. Alitta claims that if p is an odd prime then $p^2 - 2$ is also an odd prime.

Which of the following values of p is a counterexample to this claim?

A 3 B 5 C 7 D 9 E 11

Solution **E**

A counterexample to the claim is an odd prime p such that $p^2 - 2$ is *not* an odd prime.

3 is an odd prime, and $3^2 - 2 = 7$ is also an odd prime. So 3 is not a counterexample.

5 is an odd prime, and $5^2 - 2 = 23$ is also an odd prime. So 5 is not a counterexample.

7 is an odd prime, and $7^2 - 2 = 47$ is also an odd prime. So 7 is not a counterexample.

9 is not an odd prime. So 9 is not a counterexample

11 is an odd prime, but $11^2 - 2 = 119 = 7 \times 17$ is not an odd prime. Therefore 11 is a counterexample.

For investigation

11.1 For each of the following statements find a counterexample.

(a) If p is a prime, then $6p + 1$ is also a prime.

(b) If p is an integer with $p > 1$ and $6p + 1$ is a prime, then p is also a prime.

(c) If p is a prime, then $3^p + 20$ is also a prime.

(d) If p is a prime, then there is another prime between p and $p + 10$.

12. For how many positive integers N is the remainder 6 when 111 is divided by N?

A 5 B 4 C 3 D 2 E 1

Solution **A**

The remainder when 111 is divided by N is 6 provided that $111 = QN + 6$, where Q is a non-negative integer and $6 < N$. In other words, N is a factor of $111 - 6$ with $6 < N$.

Now $111 - 6 = 105$. The prime factorization of 105 is $3 \times 5 \times 7$. Therefore the factors of $111 - 6$ are 1, 3, 5, 7, 15, 21, 35 and 105.

Of these 8 factors all but 1, 3 and 5 are greater than 6.

Therefore there are 5 positive integers N which give a remainder 6 when 111 is divided by N.

For investigation

12.1 For how many positive integers N is the remainder 7 when 112 is divided by N?

13. Which of these is the mean of the other four?

A $\sqrt{2}$ B $\sqrt{18}$ C $\sqrt{200}$ D $\sqrt{32}$ E $\sqrt{8}$

SOLUTION **D**

We use the fact that one of the options is the mean of the other four options provided that it is the mean of all five of the options. Your are asked to check this fact in Problem 13.2.

The mean of the five numbers given as options is

$$\begin{aligned}\frac{\sqrt{2}+\sqrt{18}+\sqrt{200}+\sqrt{32}+\sqrt{8}}{5} &= \frac{\sqrt{2}+3\sqrt{2}+10\sqrt{2}+4\sqrt{2}+2\sqrt{2}}{5}\\ &= \left(\frac{1+3+10+4+2}{5}\right)\sqrt{2}\\ &= \left(\frac{20}{5}\right)\sqrt{2}\\ &= 4\sqrt{2}\\ &= \sqrt{32}.\end{aligned}$$

Therefore the correct option is D.

FOR INVESTIGATION

13.1 (a) Find the mean of the primes 5, 7, 11, 13 and 19.

(b) Hence show that this mean is one of these primes.

(c) Check that this mean is also the mean of the other four primes.

13.2 (a) Show that if the number p is the mean of the five numbers p, q, r, s and t, then p is also the mean of the four numbers q, r, s and t.

(b) Show that if the number p is the mean of the four numbers q, r, s and t, then p is also the mean of the five numbers p, q, r, s and t.

13.3 Show that the result of Problem 13.2 generalizes to the case of a set of n numbers, for each integer n with $n \geq 2$. That is, show that given a set of n numbers, a number p in the set is the mean of the other $n-1$ numbers in the set if and only if it is the mean of all the n numbers in the set.

13.4 Which of the seven primes consecutive 7, 11, 13, 17, 19, 23 and 27 is the mean of the other six primes in the list?

13.5 Which of the seven consecutive primes 101, 103, 107, 109, 113, 127 and 131 is the mean of the other six primes in the list?

14. What is the smallest number of rectangles, each measuring 2 cm by 3 cm, which are needed to fit together without overlap to form a rectangle whose sides are in the ratio 5 : 4 ?

A 10 B 15 C 20 D 30 E 60

Solution **D**

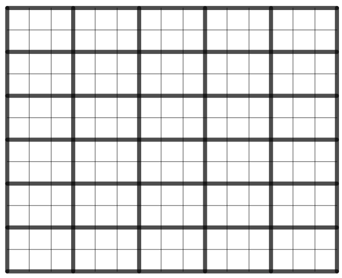
A rectangle whose sides are in the ratio 5 : 4 has dimensions $5k$ cm $\times$ $4k$ cm, for some positive number k. Since we aim to cover this rectangle with $2\,\text{cm} \times 3\,\text{cm}$ rectangles without overlap, k needs to be a positive integer.

The area of the $5k$ cm$\times$$4k$ cm rectangle is $(5k \times 4k)\,\text{cm}^2 = 20k^2\,\text{cm}^2$. The $2\,\text{cm}\times3\,\text{cm}$ rectangles have area $6\,\text{cm}^2$.

It follows that $20k^2$ needs to be a multiple of 6. The least positive integer k for which this is the case is 3. In this case the larger rectangle has dimensions $15\,\text{cm} \times 12\,\text{cm}$ and area $(15 \times 12)\,\text{cm}^2 = 180\,\text{cm}^2$. Therefore the smallest number of $2\,\text{cm} \times 3\,\text{cm}$ rectangles that are needed would be $\dfrac{180}{6} = 30$, provided that it is possible to cover a $15\,\text{cm} \times 12\,\text{cm}$ with 30 $2\,\text{cm} \times 3\,\text{cm}$ rectangles.

To complete the question we need to show that this is possible.

One way in which this can be done is shown in the diagram on the right.



Therefore the smallest number of $2\,\text{cm} \times 3\,\text{cm}$ rectangles that are needed is 30.

For investigation

14.1 Find other ways to fit together 30 rectangles measuring $2\,\text{cm}\times3\,\text{cm}$ to make a $15\,\text{cm}\times12\,\text{cm}$ rectangle.

15. Three dice, each showing numbers 1 to 6 are coloured red, blue and yellow respectively. Each of the dice is rolled once. The total of the numbers rolled is 10. In how many different ways can this happen?

A 36 B 30 C 27 D 24 E 21

Solution **C**

We note first that, because the dice are coloured, two outcomes with total 10, but with different numbers rolled on particular dice, count as being different. For example

red : 6 blue : 3 yellow : 1

counts as being different from the outcome

red : 6 blue : 1 yellow : 3.

With three different numbers there are three choices for the dice which rolls the first number, then two choices for the dice which rolls the second number, leaving just one choice for the dice which rolls the third number. This gives a total of $3 \times 2 \times 1 = 6$ arrangements for the three numbers.

It can be checked that when two of the numbers are the same these can occur in 3 different ways.

It is not possible to have three equal scores with total 10.

To solve this problem we now list in the following table all possible ways a total of 10 may be obtained by throwing three dice. In each row of the table we also give the number of different ways the three numbers in the row may be arranged between the three dice.

scores	no. of ways
6, 3, 1	6
6, 2, 2	3
5, 4, 1	6
5, 3, 2	6
4, 4, 2	3
4, 3, 3	3

Therefore the total number of different ways of achieving a total of 10 is $6 + 3 + 6 + 6 + 3 + 3 = 27$.

For investigation

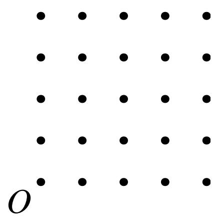
15.1 Check that when two of the numbers are the same they can occur in 3 different ways on the three dice.

15.2 In how many different ways can the total of the numbers rolled be 12?

15.3 For $3 \leq T \leq 18$, calculate the number of different ways in which a total of T can be rolled using the three dice.

What do you notice about the answers?

16. An array of 25 equally spaced dots is drawn in a square grid as shown. Point O is in the bottom left corner. Linda wants to draw a straight line through the diagram which passes through O and exactly one other point.

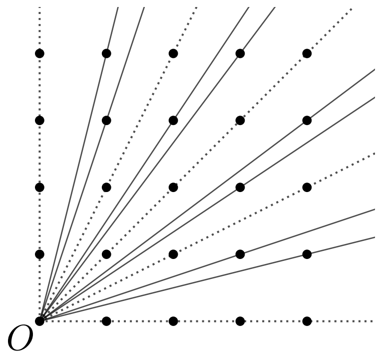


How many such lines can Linda draw?

A 4 B 6 C 8 D 12 E 24

Solution **C**

In the diagram on the right the solid lines go through O and exactly one other point, and the dotted lines go through O and at least two other points.



There is a line through every point so all possible lines have been considered.

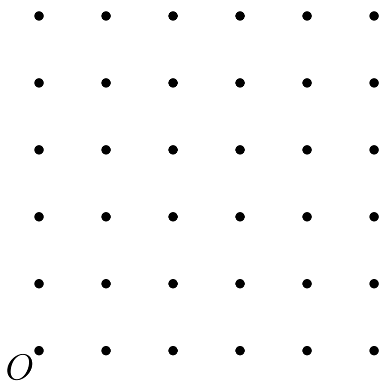
The solid lines are the lines that Linda can draw. We therefore see that the number of lines that Linda can draw is 8.

[*Note*: Because the diagram is symmetric about the bottom-left to top-right diagonal, it was only really necessary to draw half the lines in the diagram.]

For investigation

16.1 An array of 36 equally spaced dots is drawn in a square grid as shown. Mollie wants to draw a straight line which passes through the dot marked O and exactly one other dot.

How many of these lines can Mollie draw?



16.2 Naomi has a piece of paper on which are drawn 400 equally spaced dots in a square 20×20 grid.

Naomi wants to draw a straight line which passes through the bottom left-hand dot and exactly one other dot.

How many of these lines can Naomi draw?

16.3 Olivia has a piece of paper on which are drawn 10 000 equally spaced dots in a square 100×100 grid.

Olivia wants to draw a straight line which passes through the bottom left-hand dot and exactly one other dot.

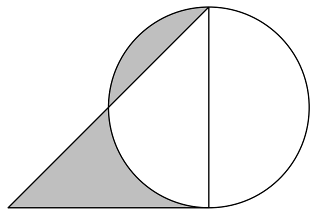
How many of these lines can Olivia draw?

17. A circle of radius r and a right-angled isosceles triangle are drawn such that one of the shorter sides of the triangle is a diameter of the circle.

What is the shaded area?

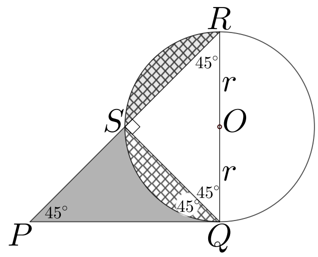
A $\sqrt{2}r$ B r^2 C $2\pi r$ D $\frac{\pi r^2}{4}$

E $(\sqrt{2} - 1)\pi r^2$



Solution **B**

Let O be the centre of the circle and let P, Q, R and S be the points as shown in the diagram.



Because PQR is a right-angled isosceles triangle, $\angle PRQ = \angle RPQ = 45°$, and $PQ = QR = 2r$.

Because the angle in a semicircle is a right angle [this is Thales' theorem], $\angle RSQ = 90°$. Therefore, because the sum of the angles in a triangle is $180°$, we have $\angle RQS = 45°$.

We therefore have $\angle SRQ = 45° = \angle SQR$. It follows that $SQ = SR$.

Because $SQ = SR$ the segments of the circle cut off by these lines, shown as hatched in the diagram, are congruent. Hence they have the same area.

It follows that the shaded area is the same as the area of the triangle PQS.

PQS is a right-angled isosceles triangle with hypotenuse PQ of length $2r$. Therefore the triangle PQS has area r^2.

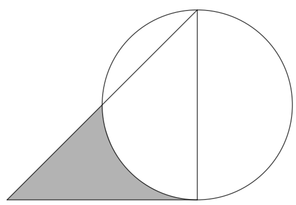
Therefore the shaded area is r^2.

For investigation

17.1 Explain why from the fact that the hypotenuse of the triangle PQS has length $2r$, it follows that the area of the triangle is r^2.

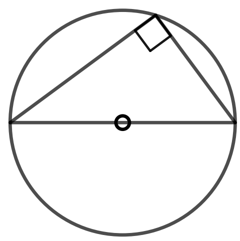
17.2 In the diagram on the right there is a circle of radius r and a right-angled isosceles triangle. One of the shorter sides of the triangle is a diameter of the circle.

What is the shaded area?



17.3 Give a proof of Thales' theorem:

The angle in a semicircle is a right angle.



18. The number 840 can be written as $\dfrac{p!}{q!}$, where p and q are positive integers less than 10.

What is the value of $p + q$?

Note that, $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.

A 8 B 9 C 10 D 12 E 15

Solution **C**

We note first that, as $\dfrac{p!}{q!} = 840$, it follows that $p! > q!$ and hence $p > q$. Therefore

$$\frac{p!}{q!} = \frac{1 \times 2 \times \cdots \times q \times (q+1) \times \cdots \times p}{1 \times 2 \times \cdots \times q} = (q+1) \times (q+2) \times \cdots \times (p-1) \times p.$$

Thus $840 = \dfrac{p!}{q!}$ is the product of the consecutive integers $q+1, q+2, \ldots, p-1, p$, where $p \leq 9$.

Since 840 is not a multiple of 9, $p \neq 9$. Since 840 is a multiple of 7, $p \geq 7$.

Now $5 \times 6 \times 7 \times 8 = 1680 > 840$, while $6 \times 7 \times 8 = 336 < 840$. Hence 840 is not the product of consecutive integers of which the largest is 8.

We deduce that $p = 7$. It is now straightforward to check that

$$840 = 4 \times 5 \times 6 \times 7 = \frac{7!}{3!}.$$

Therefore $p = 7$ and $q = 3$. Hence $p + q = 7 + 3 = 10$.

For investigation

18.1 Find positive integers p and q with $q < p \leq 20$ such that

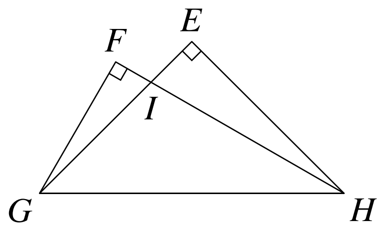
$$\frac{p!}{q!} = 2730.$$

18.2 Is it possible to find positive integers p and q with $q < p \leq 20$ such that

$$\frac{p!}{q!} = 253\,?$$

19. The diagram shows two overlapping triangles: triangle FGH with interior angles $60°$, $30°$ and $90°$ and triangle EGH which is a right-angled isosceles triangle.

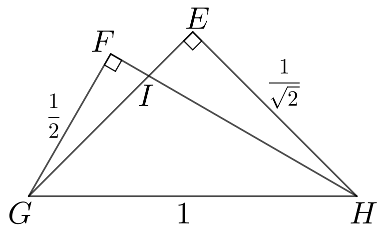
What is the ratio of the area of triangle IFG to the area of triangle IEH?



A $1:1$ B $1:\sqrt{2}$ C $1:\sqrt{3}$ D $1:2$ E $1:3$

Solution **D**

We suppose that we have chosen units so that the length of GH is 1.



Because FGH is a $60°, 30°, 90°$ triangle, it follows that FG has length $\frac{1}{2}$.

Because EGH is a right-angled isosceles triangle it also follows that EH has length $\frac{1}{\sqrt{2}}$.

In the triangles IFG and IEH we have

$$\angle GFI = \angle HEI = 90°$$

and

$$\angle GIF = \angle HIE \text{ (vertically opposite angles).}$$

Because the sum of the angles in both these triangles is $180°$, it follows that

$$\angle FGI = \angle EHI.$$

Therefore the triangles IFG and IEH are similar.

The ratio of the areas of similar triangles equals the ratio of the squares of the lengths of corresponding sides. Therefore

$$\begin{aligned}\text{area of } IFG : \text{area of } IEH &= FG^2 : EH^2 \\ &= \left(\tfrac{1}{2}\right)^2 : \left(\tfrac{1}{\sqrt{2}}\right)^2 \\ &= \tfrac{1}{4} : \tfrac{1}{2} \\ &= 1 : 2.\end{aligned}$$

For investigation

19.1 Explain why, given that GH has length 1, FG has length $\frac{1}{2}$ and EH has length $\frac{1}{\sqrt{2}}$.

19.2 Explain why the ratio of the areas of similar triangles equals the ratio of the squares of the lengths of corresponding sides.

19.3 (a) Given that GH has length 1, find the area of the triangle GIH.

(b) Given that GH has length 1, find the areas of the triangles IFG and IEH.

(c) Hence verify that the ratio of the areas of the triangles IFG and IEH is $1:2$.

20. Laura and Dina have a running race. Laura runs at constant speed and Dina runs n times as fast where $n > 1$. Laura starts s m in front of Dina.

What distance, in metres, does Dina run before she overtakes Laura?

A $\dfrac{ns}{n-1}$ B ns C $\dfrac{s}{n-1}$ D $\dfrac{ns}{n+1}$ E $\dfrac{s}{n}$

Solution **A**

Suppose that Dina has run a distance of d metres when she overtakes Laura. Because Laura has a start of s metres, at this time Laura has run a distance of $d - s$ metres.

Because they have been running for the same amount of time when Dina overtakes Laura, at this time the ratio of the distances they have run is the same as the ratio of their speeds. That is

$$d : d - s = n : 1.$$

It follows that

$$\frac{d}{d-s} = \frac{n}{1}.$$

Hence

$$d = n(d - s).$$

This last equation may be rearranged as

$$ns = d(n - 1).$$

Therefore

$$d = \frac{ns}{n-1}.$$

For investigation

20.1 Suppose that when Dina overtakes Laura she has run twice as far as Laura.

What is the ratio of Dina's speed to Laura's speed?

21. The numbers m and k satisfy the equations $2^m + 2^k = p$ and $2^m - 2^k = q$.

What is the value of 2^{m+k} in terms of p and q?

A $\dfrac{p^2 - q^2}{4}$ B $\dfrac{pq}{2}$ C $p + q$ D $\dfrac{(p-q)^2}{4}$ E $\dfrac{p+q}{p-q}$

SOLUTION **A**

We have

$$2^m + 2^k = p \qquad (1)$$

and

$$2^m - 2^k = q. \qquad (2)$$

Adding equations (1) and (2), we obtain

$$2(2^m) = p + q$$

and hence

$$2^m = \frac{p+q}{2}. \qquad (3)$$

Subtracting equations (2) from equation (1), we obtain

$$2(2^k) = p - q$$

and hence

$$2^k = \frac{p-q}{2}. \qquad (4)$$

Therefore, by (3) and (4)

$$\begin{aligned} 2^{m+k} &= 2^m \times 2^k \\ &= \left(\frac{p+q}{2}\right) \times \left(\frac{p-q}{2}\right) \\ &= \frac{(p+q)(p-q)}{4} \\ &= \frac{p^2 - q^2}{4}. \end{aligned}$$

FOR INVESTIGATION

21.1 Use the equations $p = 2^m + 2^k$ and $q = 2^m - 2^k$ to obtain expressions for p^2 and q^2.

Hence deduce that $\dfrac{p^2 - q^2}{4} = 2^{m+k}$.

21.2 The numbers a and b satisfy the equations

$$2^{a+b} = r$$

and

$$2^{a-b} = s.$$

Find $2^a + 2^b$ in terms of r and s.

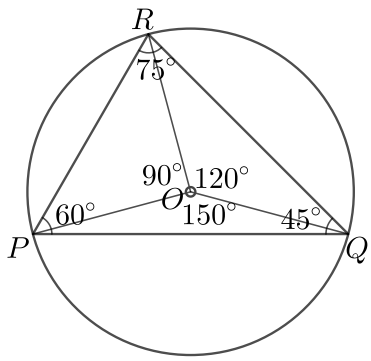
22. A triangle with interior angles 60°, 45° and 75° is inscribed in a circle of radius 2.

What is the area of the triangle?

A $2\sqrt{3}$ B 4 C $6+\sqrt{3}$ D $6\sqrt{3}$ E $3+\sqrt{3}$

Solution **E**

Let P, Q and R be the vertices of the triangle, let O be the centre of the circle, as shown in the diagram.



The angle subtended by an arc at the centre of a circle is twice the angle subtended at the circumference. Therefore

$$\angle QOR = 120°,$$

$$\angle ROP = 90°$$

and

$$\angle POQ = 150°.$$

We use the notation $[XYZ]$ for the area of a triangle XYZ.

Using the formula $\frac{1}{2}ab\sin\theta$ for the area of a triangle with sides of lengths a and b with included angle θ, we have

$$[QOR] = \frac{1}{2}(OQ \times OR)\sin 120° = \frac{1}{2}(2\times 2)\frac{\sqrt{3}}{2} = \sqrt{3},$$

$$[ROP] = \frac{1}{2}(OR \times OP)\sin 90° = \frac{1}{2}(2\times 2) = 2$$

and

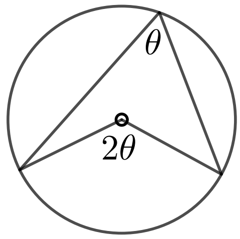
$$[POQ] = \frac{1}{2}(OP \times OQ)\sin 150° = \frac{1}{2}(2\times 2)\frac{1}{2} = 1.$$

Therefore

$$[PQR] = [QOR] + [ROP] + [POQ] = \sqrt{3} + 2 + 1 = 3 + \sqrt{3}.$$

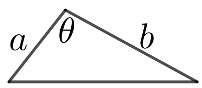
For investigation

22.1 Find a proof that the angle subtended by an arc at the centre of a circle is twice the angle subtended at the circumference. [That is, try and prove this for yourself, or find a proof in a book or on the internet, or ask your teacher.]



22.2 Show that $\sin 120° = \dfrac{\sqrt{3}}{2}$ and $\sin 150° = \dfrac{1}{2}$.

22.3 Show that the area of a triangle with side lengths a and b with included angle θ is $\frac{1}{2}ab\sin\theta$.



23. Let x be a real number. What is the minimum value of $\left(x^2 - 4x + 3\right)\left(x^2 + 4x + 3\right)$?

A -16 B -9 C 0 D 9 E 16

Solution **A**

We have

$$\begin{aligned}\left(x^2 - 4x + 3\right)\left(x^2 + 4x + 3\right) &= \left((x^2+3) - 4x\right)\left((x^2+3) + 4x\right)\\ &= (x^2+3)^2 - (4x)^2\\ &= x^4 + 6x^2 + 9 - 16x^2\\ &= x^4 - 10x^2 + 9\\ &= (x^2-5)^2 - 16.\end{aligned}$$

For every real number x, $(x^2-5)^2 \geq 0$ and therefore $(x^2-5)^2 - 16 \geq -16$.

Now, when $x = \sqrt{5}$, $(x^2-5)^2 - 16 = 0^2 - 16 = -16$.

It follows that the minimum value of $\left(x^2 - 4x + 3\right)\left(x^2 + 4x + 3\right)$ is -16.

For investigation

23.1 (a) Find the real numbers a and b for which

$$x^4 - 8x^2 + 12 = (x^2 + a)^2 + b, \text{ for all real numbers } x.$$

(b) Hence find the minimum value of $x^4 - 8x^2 + 12$.

23.2 (a) Find the real numbers a and b for which

$$x^4 + 8x^2 + 12 = (x^2 + a)^2 + b, \text{ for all real numbers } x.$$

(b) Hence find the minimum value of $x^4 + 8x^2 + 12$.

Note

If you have met the *differential calculus*, you will know that the minimum values of polynomials may be found using calculus. In this case, check that using calculus you obtain the minimum value -16 for the polynomial $(x^2 - 4x + 3)(x^2 + 4x + 3)$.

Also, use calculus to solve Problems 23.1 and 23.2.

24. Saba, Rayan and Derin are cooperating to complete a task. They each work at a constant rate independent of whoever else is working on the task. When all three work together, it takes 5 minutes to complete the task. When Saba is working with Derin, the task takes 7 minutes to complete. When Rayan is working with Derin, the task takes 15 minutes to complete.

How many minutes does it take for Derin to complete the task on his own?

A 21 B 28 C 35 D 48 E 105

Solution **E**

Suppose that it takes Saba, Rayan and Derin, working on their own, s, r and d minutes, respectively, to complete the task.

Then in 1 minute Saba completes $\frac{1}{s}$ of the task, Rayan completes $\frac{1}{r}$ of the task, and Derin completes $\frac{1}{d}$ of the task. Hence when all three are working together in 1 minute they complete $\frac{1}{s}+\frac{1}{r}+\frac{1}{d}$ of the task. Since it takes them 5 minutes to complete the task when they all work together,

$$\frac{1}{s}+\frac{1}{r}+\frac{1}{d}=\frac{1}{5}.$$

Similarly, as it takes Saba working with Derin 7 minutes to complete the task,

$$\frac{1}{s}+\frac{1}{d}=\frac{1}{7}.$$

Likewise, as it takes Rayan working with Derin 15 minutes to complete the task,

$$\frac{1}{r}+\frac{1}{d}=\frac{1}{15}.$$

We therefore have

$$\begin{aligned}\frac{1}{d}&=\left(\frac{1}{s}+\frac{1}{d}\right)+\left(\frac{1}{r}+\frac{1}{d}\right)-\left(\frac{1}{s}+\frac{1}{r}+\frac{1}{d}\right)\\&=\frac{1}{7}+\frac{1}{15}-\frac{1}{5}\\&=\frac{15}{105}+\frac{7}{105}-\frac{21}{105}\\&=\frac{1}{105}.\end{aligned}$$

Therefore it takes Derin 105 minutes to complete the task on his own.

For investigation

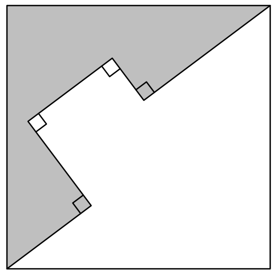
24.1 (a) How many minutes does it take for Rayan to complete the task on his own?

(b) How many minutes does it take for Saba to complete the task on her own?

25. Five line segments of length 2, 2, 2, 1 and 3 connect two corners of a square as shown in the diagram.

What is the shaded area?

A 8 B 9 C 10 D 11 E 12

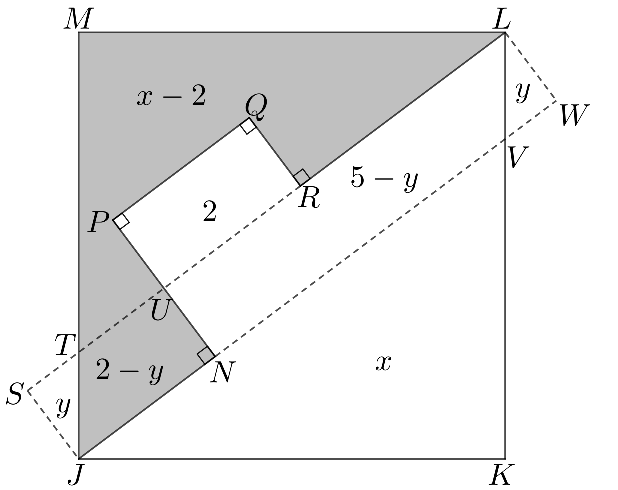


Solution **B**

Commentary

There are many different ways in which this problem may be solved. We give a solution which involves few calculations, but quite a lot of facts about the diagram. You are asked to check these facts in Problems 25.1 and 25.2.

Three other ways of solving the problem are indicated in problems 25.3, 25.4 and 25.5.



We let the points in the diagram be labelled as shown. The points S and W are chosen so that $SJWL$ is a rectangle.

The shaded region is the polygon $JNPQRLM$. We use the notation $[JNPQRLM]$ for the area of this polygon, and similar notation for the areas of other polygons.

In the rectangle $SJWL$ we have $JW = 7$ and $WL = 1$. Therefore, by Pythagoras' Theorem applied to the triangle JWL, we have $JL^2 = 7^2 + 1^2 = 50$. Hence $JL = \sqrt{50} = 5\sqrt{2}$. Since the diagonal of the square $JKLM$ has length $5\sqrt{2}$, it follows that the side length of the square is 5. Hence $[JKLM] = 5^2 = 25$. (You are asked to check all these facts in Problem 25.1.)

The triangles JKV and LMT are congruent. We let x be the common area of these triangles. Also, the triangles JTS and LVW are congruent. We let y be the common area of these triangles. It follows that the areas of the polygons in the diagram are as shown. (You are asked to check all these facts in Problem 25.2.)

We have

$$[JKLM] = [JKV] + [LMT] + [SJWL] - [JTS] - [LVW],$$

and therefore

$$25 = 2x + 7 - 2y.$$

Hence

$$2x - 2y = 25 - 7 = 18,$$

and therefore

$$x - y = 9.$$

It follows that

$$[JNPQRLM] = [TUPQRLM] + [JNUT] = (x - 2) + (2 - y) = x - y = 9.$$

Note: In the problems below, we use the same notation as in the solution above.

FOR INVESTIGATION

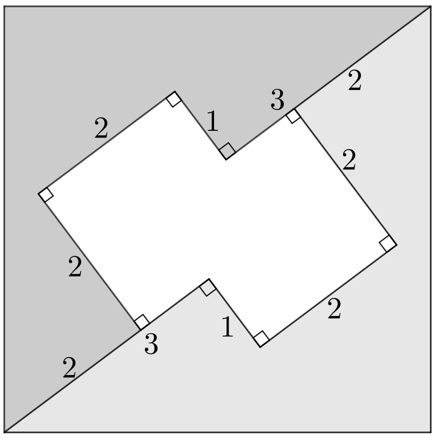
25.1 (a) Explain why in the rectangle $SJWL$ we have $JW = 7$ and $WL = 1$, and hence $JL = 5\sqrt{2}$.

(b) Explain why it follows from the fact that the diagonal JL has length $5\sqrt{2}$ that the square $JKLM$ has side length 5.

25.2 (a) Show the the triangles JKV and LMT are congruent, and that the triangles JTS and LVW are congruent.

(b) Deduce that the areas of the polygons in the diagram are as shown.

25.3



The lightly shaded region in the diagram above has been drawn so that it is congruent to the region whose area we need to find.

(a) Find the area of the unshaded region in the diagram.

(b) Use the fact that the area of the square is 25 to find the area of each of the shaded regions.

25.4 (a) Show that the triangles JKV and LWV are similar.

(b) Deduce that LV has length $\frac{5}{4}$ and that VK has length $\frac{15}{4}$.

(c) Use these lengths to find the values of x and y.

Hence check that $x - y = 9$.

25.5 Let $\angle VJK = \theta$.

(a) By considering the projections of the line segments JN, NP, PQ, QR and RL on JK and on JM, show that both

$$7\cos\theta - \sin\theta = JK$$

and

$$7\sin\theta + \cos\theta = JM.$$

(b) Use the fact that $JK = JM$ to deduce that $\sin\theta = \frac{3}{4}\cos\theta$. Hence find the values of $\cos\theta$ and $\sin\theta$.

(c) Use the values of $\cos\theta$ and $\sin\theta$ to calculate the area of the polygon $JNPQRLM$.