United Kingdom
Mathematics Trust

Senior Mathematical Challenge

Thursday 11 November 2021

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For reasons of space, these solutions are necessarily brief.

There are more in-depth, extended solutions available on the UKMT website, which include some exercises for further investigation.

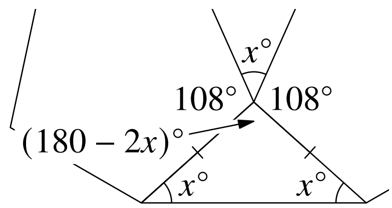
There is also a version of this document available on the UKMT website which includes each of the questions alongside its solution:

www.ukmt.org.uk

1. C Since Cicely was 21 in 1939, she was born in 1918 as $1939 - 21 = 1918$. So her 100^{th} birthday would have been in 2018.

2. C Terms of 40 in this sequence come from all primes in the interval 35 to 44 inclusive, namely 37, 41 and 43. So three terms in this sequence are equal to 40.

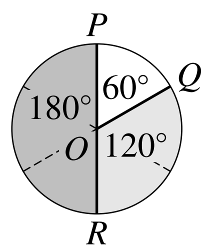
3. C Both pentagons are regular and congruent so the triangle is isosceles, with angles $x°$, $x°$ and $(180 - 2x)°$. The interior angles of the pentagons are 108°. Considering angles around the point where the pentagons meet, we have $108° + x° + 108° + (180 - 2x)° = 360°$, which simplifies to $396° - x° = 360°$ and therefore $x = 36$.



4. D Re-writing the equation as $\dfrac{\frac{k}{12}}{\frac{15}{k}} = 20$ and then rearranging gives $\frac{k}{12} = 20 \times \frac{15}{k}$ and so $k^2 = 3600$. As $k > 0$, $k = 60$ so the sum of the digits of k is 6.

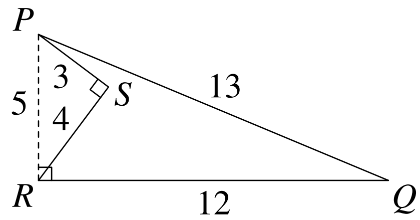
5. E For the sum of four primes to itself be prime, that sum must be odd. This means that one of the primes must be even. The four consecutive primes can only be 2, 3, 5 and 7 and their sum, 17, is indeed a prime. The largest of those four primes is 7.

6. B The arc lengths PQ, QR and RP being in the ratio $1 : 2 : 3$ corresponds to the angles at point O also being in the ratio $1 : 2 : 3$, so the angles are 60°, 120° and 180° as shown. The ratio of the areas of sectors OQ, QOR and ROP is then also $1 : 2 : 3$.



7. **B** Repeatedly using the rule of indices $a^{bc} = (a^b)^c$ allows us to express each option in the form $(a^b)^{1000}$. Thus $2^{5000} = (2^5)^{1000} = 32^{1000}$, $3^{4000} = (3^4)^{1000} = 81^{1000}$, $4^{3000} = (4^3)^{1000} = 64^{1000}$ and $5^{2000} = (5^2)^{1000} = 25^{1000}$. The last option, 6^{1000}, is already in the required form. Comparing the base numbers 32, 81, 64, 25 and 6 allows us to see that the largest of the numbers given is $81^{1000} = 3^{4000}$.

8. **B** Joining P to R forms the hypotenuse of right-angled triangle PSR. Using Pythagoras' Theorem, $PR = \sqrt{3^2 + 4^2} = 5$. Considering triangle PRQ and noting that $5^2 + 12^2 = 13^2$ implies, by the converse of Pythagoras' Theorem, that PR is perpendicular to RQ. So the area of the region is $\frac{5\times 12}{2} - \frac{3\times 4}{2} = 30 - 6 = 24$.



9. **D** The numbers 0 and 5 must be paired with one another (one choice). The number 1 must be paired with either 4 or 9 (two choices). The number 6 must be paired with whichever of 4 or 9 is not paired with 1 (one choice). Similarly, the number 2 must be paired with either 3 or 8 (two choices). Lastly, the number 7 must be paired with whichever of 3 or 8 is not paired with 2 (one choice) . The number of possible combinations is then the product of the number of choices which is $1 \times 2 \times 1 \times 2 \times 1 = 4$.

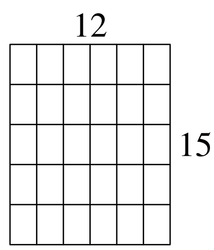
10. **D** To find the smallest number of people who could have been surveyed, we can find the largest angle that can be used to represent one person's answer in the survey. This angle, in degrees, is the highest common factor of 40, 72, 108 and 140 which is 4. If each person's answer is represented by an angle of $4°$, the number of people surveyed is $\frac{360°}{4°} = 90$.

11. **E** Consider the five options in turn: $3^2 - 2 = 9 - 2 = 7$ which is prime; $5^2 - 2 = 25 - 2 = 23$ which is prime; $7^2 - 2 = 49 - 2 = 47$ which is prime; 9 is not a prime; and $11^2 - 2 = 121 - 2 = 119 = 7 \times 17$. So option E provides a counterexample.

12. **A** In order to have a remainder of 6 when 111 is divided by the positive integer N we must be able to express 111 as $111 = k \times N + 6$ for some positive integer k. Equivalently, $105 = k \times N$ and so N is a factor of 105. The prime factorisation of 105 is $3 \times 5 \times 7$ and so the factors of 105 are 1, 3, 5, 7, 15, 21, 35 and 105. To produce a remainder of 6, the divisor N must be greater than 6 so there are five possible values of N: 7, 15, 21, 35 and 105.

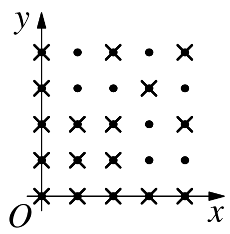
13. **D** Asking which of the five numbers is the mean of the other four is equivalent to asking for the mean of all five numbers. Each of the options can be rewritten as a multiple of $\sqrt{2}$. $A = \sqrt{2} = 1 \times \sqrt{2}$, $B = \sqrt{18} = 3 \times \sqrt{2}$, $C = \sqrt{200} = 10 \times \sqrt{2}$, $D = \sqrt{32} = 4 \times \sqrt{2}$ and $E = \sqrt{8} = 2 \times \sqrt{2}$. Finding the mean of 1, 3, 10, 4 and 2 gives $\frac{1+3+10+4+2}{5} = \frac{20}{5} = 4$. Therefore the mean is $4\sqrt{2}$ which is $\sqrt{32}$.

14. **D** Consider the areas of the small rectangles and the larger rectangle. Each small, 2 cm by 3 cm, rectangle has area $6\,\text{cm}^2$. The larger rectangle has sides in the ratio 4 : 5 so has sides of length $4k$ and $5k$, for some positive integer k, giving an area of $20k^2$. Values of $20k^2$ are 20, 80, 180, 320, ... The smallest of these which is a multiple of 6 is 180, when $k = 3$. The sides of the larger rectangle are then 12 cm and 15 cm. Now we can check to see that the $\frac{180\,\text{cm}^2}{6\,\text{cm}^2} = 30$ small rectangles can be arranged. One possible arrangement of five rows containing six small rectangles is as shown.



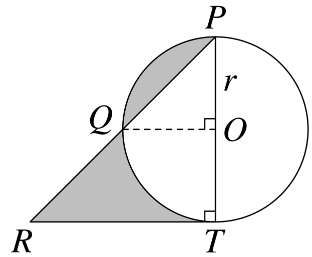
15. C To reach a total of 10, each of the dice could show a different number. Alternatively, two of the dice could show the same numbers. However, it is not possible that all three numbers could be the same, as 10 is not a multiple of three. The possible sets of three different numbers are (1, 3, 6), (1, 4, 5) and (2, 3, 5) and there are six ways each could come from the three coloured dice. The sets which include a repeated number are (2, 2, 6), (3, 3, 4) and (4, 4, 2) and there are three ways each of these could come from the coloured dice. In total this gives $3 \times 6 + 3 \times 3 = 27$ ways.

16. C Overlaying the 25 dots onto a coordinate grid allows us to describe their positions using (x, y). Point O is at $(0, 0)$. Lines through O and any points where either $x = 0$ or $y = 0$ or $y = x$ are not possible as these lines will pass though more than the two allowable points. Above the line $y = x$, Linda can choose to draw lines through O and one of $(1, 3)$, $(1, 4)$, $(2, 3)$ or $(3, 4)$, but not through O and either $(1, 2)$ or $(2, 4)$ as those two points both lie on the same line, $y = 2x$.



By symmetry, there are also four valid points under the line $y = x$ she could choose through which to draw her line, by exchanging the x and y coordinates. This makes a total of eight lines that could be drawn fitting the given criteria.

17. B In the right-angled isosceles triangle PRT, $PT = 2r = RT$. As $\angle OPQ = 45°$ and triangle OPQ is isoceles, $\angle POQ = 90°$. Therefore the segment above line PQ can be reflected in a line through O and Q to become a segment below QT which then completes a new shaded triangle, QRT. Triangle QRT has base RT of length $2r$ and perpendicular height OT, of length r. The shaded area is then $\frac{2r \times r}{2} = r^2$.

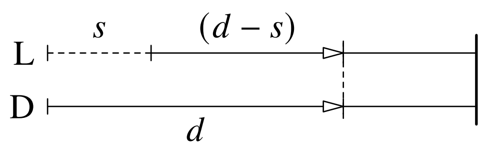


18. C When written as the product of its prime factors, $840 = 2 \times 2 \times 2 \times 3 \times 5 \times 7$. Rearranging the order of the primes gives $840 = 7 \times 3 \times 2 \times 5 \times 2 \times 2 = 7 \times 6 \times 5 \times 4 = 7 \times 6 \times 5 \times 4 \times \frac{3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{7!}{3!}$. Therefore $p = 7$ and $q = 3$, so $p + q = 10$. One can check that there are no further possible solutions with p and q less than 10.

19. D We consider triangles IFG and IEH. Angles GFI and HEI are right angles and angles GIF and HIE are equal since they are vertically opposite. Hence the angles FGI and EHI are equal using 'the angle sum of a triangle is 180°'. Triangles IFG and IEH have the same angles and therefore are similar.
Let FG be of length x. Then GH has length $2x$ as triangle FGH is a 30°, 60°, 90° triangle. Now considering triangle EGH, the length of EH is $\frac{2x}{\sqrt{2}} = \sqrt{2}x$ as this is a 45°, 45°, 90° triangle.

Comparing the corresponding lengths of FG and EH we have a length scale factor of $\sqrt{2}$ and so comparing the areas of triangles IFG and IEH we have an area scale factor of $(\sqrt{2})^2$. So the ratio of the areas is 1 : 2.

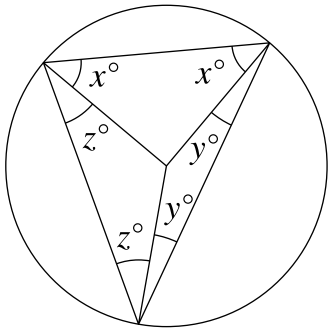
20. A Let Laura's speed be v m/s, so Dina's speed is nv m/s. Let the distance Dina runs before she overtakes Laura be d m and the time until that happens be t s.

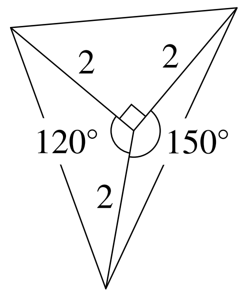


Using distance = speed × time, for Laura we have $d - s = vt$ and for Dina we have $d = nvt$. Eliminating v and t between these two equations allows us to find d in terms of n and s, so $d = n(d - s)$. This expands and rearranges to $nd - d = ns$ and so $d(n - 1) = ns$. Dividing through gives $d = \frac{ns}{n-1}$.

21. A Adding the equations $2^m + 2^k = p$ and $2^m - 2^k = q$, gives $2 \times 2^m = p + q$. Therefore $2^m = \frac{p+q}{2}$. Similarly, subtracting the equations gives $2 \times 2^k = p - q$. Therefore $2^k = \frac{p-q}{2}$. As 2^{m+k} is $2^m \times 2^k$, $2^{m+k} = \frac{(p+q)}{2} \times \frac{(p-q)}{2} = \frac{(p^2-q^2)}{4}$.

22. E First split the inscribed triangle into three isosceles triangles, each with two vertices on the circle and the third vertex at the centre of the circle. Let the base angles in the isosceles triangles be $x°$, $y°$ and $z°$ as shown here, where $x + y = 60$, $y + z = 45$ and $x + z = 75$. Adding the equations gives $2(x + y + z) = 180$ and therefore $x + y + z = 90$. Subtracting the earlier equations from this, one at a time, gives $z = 30$, $x = 45$, and $y = 15$.



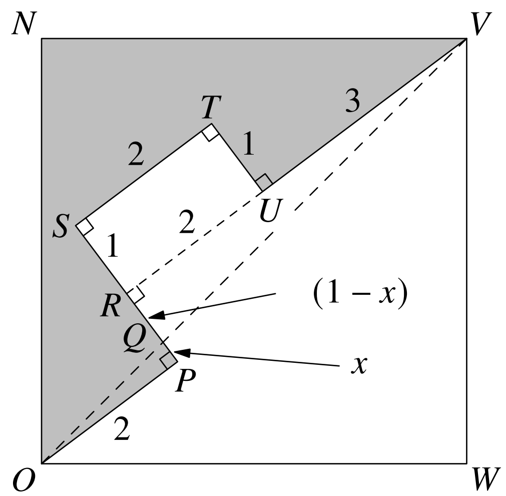


The angles of the isosceles triangles at the centre of the circle are then 150°, 120°, and 90°. These angles could also have been found using 'the angle at the centre is twice the angle at the circumference'. Using the formula 'area $= \frac{1}{2}ab\sin C$', with $a = b = 2$, gives the total area as $\frac{1}{2} \times 2 \times 2 \times \sin 150° + \frac{1}{2} \times 2 \times 2 \times \sin 120° + \frac{1}{2} \times 2 \times 2$. Since $\sin 150° = \sin 30° = \frac{1}{2}$ and $\sin 120° = \sin 60° = \frac{\sqrt{3}}{2}$, the total area $= (2 \times \frac{1}{2}) + (2 \times \frac{\sqrt{3}}{2}) + 2 = 1 + \sqrt{3} + 2 = 3 + \sqrt{3}$.

23. A Expanding the expression gives $\left(x^2 - 4x + 3\right)\left(x^2 + 4x + 3\right) = x^4 - 10x^2 + 9$ which in completed square form is $\left(x^2 - 5\right)^2 - 16$. This expression has a minimum when $x^2 - 5 = 0$ and that minimum value is then $0^2 - 16 = -16$.

24. E Let Derin work at d units per minute, let Saba work at s units per minute and let Rayan work at r units per minute. Let there be w units of work to be done to complete the task. Derin's time to complete the task is $\frac{w}{d}$. When all three work together we have (1) $w = 5d + 5s + 5r$. When Saba and Derin work together we have (2) $w = 7d + 7s$. Finally, when Rayan and Derin work together we have (3) $w = 15d + 15r$. Combining (1) and (3) to eliminate d and r gives $2w = 15s$ so $w = \frac{15s}{2}$. Combining this with (2) gives $\frac{s}{2} = 7d$ so $d = \frac{s}{14}$. Therefore $\frac{w}{d} = \frac{15s}{2} \times \frac{14}{s} = 15 \times 7 = 105$. Hence Derin would take 105 minutes.

25. B By identifying similar right-angled triangles, we can first calculate the side-length of the large square. Drawing an extra line RU to complete rectangle $RSTU$ gives $SR = 1$ and $RV = 5$. A straight line from O to V passes through SP at Q. Let $PQ = x$ and therefore $QR = 1 - x$. As $\angle OQP$ and $\angle VQR$ are vertically opposite, they are equal, so triangle OQP and triangle VQR are similar. Therefore $\frac{x}{2} = \frac{1-x}{5}$ which rearranges to give $x = \frac{2}{7}$. The ratio $PQ : QR = 2 : 5$ and so the ratio $OQ : QV = 2 : 5$. This gives $OV = \frac{7}{2} \times OQ$. Using Pythagoras' Theorem, $OV = \frac{7}{2} \times \sqrt{2^2 + (\frac{2}{7})^2} = 5\sqrt{2}$. So $OW = VW = 5$.



The shaded area is then area of triangle VNO − area of rectangle $RSTU$ − area of triangle VRQ + area of triangle $OPQ = \frac{1}{2} \times 5 \times 5 - 1 \times 2 - \frac{1}{2} \times (2 + 3) \times \frac{5}{7} + \frac{1}{2} \times 2 \times \frac{2}{7} = 9$.