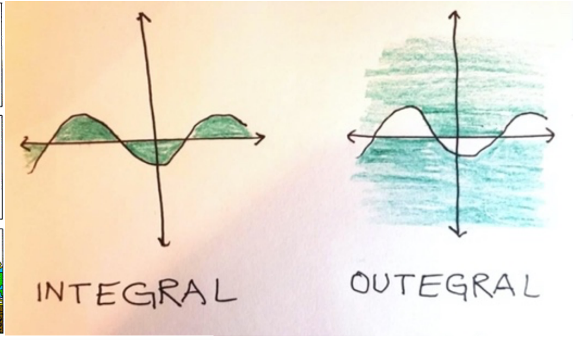
Integration: Finding Areas Notes



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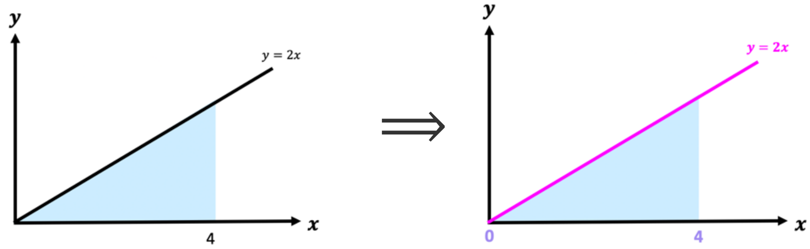
1 Introduction

Integration is used to find the area of a region bounded by a lines and curves. To find the area we just integrate the equation of the line or curve with the necessary limits!

$$\int_a^b \text{equation } dx$$

Don't worry about the dx part. That just tells us that the variable we are using in the equation is x. You should already be familiar with this from when you first learnt integration (see my integration basics rules notes if not).

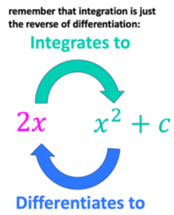
For example, if we want to find the blue area under the line of $y = 2x$ between $x = 0$ and $x = 4$:



We simply integrate the equation $2x$ and use the limits 0 and 4 since the blue shaded region corresponds to these values on the x axis.

$$\int_0^4 2x\,dx$$

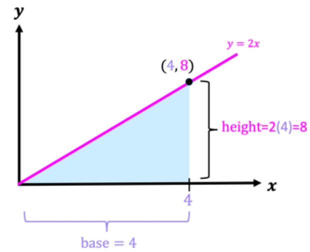
Now we can work out this integral easily by integrating (recall that the rule is to add one to the power and divide by this to integrate) and plugging in the limits. We don't need the $+c$ when we have limits (see integration basic rules notes if you struggle with this).



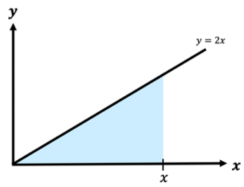
$$= [x^2]_0^4 = 4^2 - 0^2 = 16 - 0 = 16$$

Let's verify this, since we know this is just the area of a triangle $= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(4)(8) = 16$

The answers match!

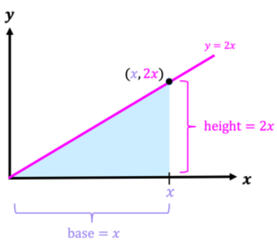


The fact that $2x$ integrates to x^2 can also be verified by calculating the area under the line $y = 2x$



We know the area of a triangle is $\frac{1}{2} \times \text{base} \times \text{height}$

Area $= \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(x)(2x) = x^2$

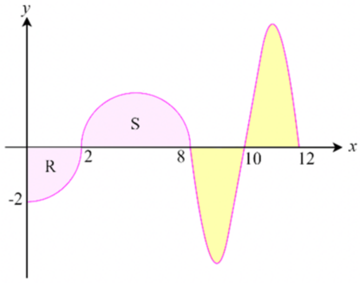


Which we know is correct since when we integrate $2x$ we get x^2. So, it should be clear that integration is the same notion as finding any area. All we have to do is set up the integral with the correct limits and integrate!

You may be wondering, why do we need integration then if we can just work out the areas using our familiar formulae for areas of shapes? Curves will not always be nice shapes such as triangles, squares, circles, rectangles, trapezium etc that we have names and formulae for. **Integration can help us find the area of any shape!**

Consider the following example. The diagram above shows the graph of a function for $0 \leq x \leq 12$. The pink shaded parts of the graph show

- A region R for $0 \leq x \leq 2$ which is a quarter circle of radius 2 with centre at the origin.
- A region S for $2 \leq x \leq 8$ which is a semicircle with diameter 6.



We can find the pink shaded area R by working out the area of a quarter circle with radius 4 and the pink shaded region S by working out the area of a semi-circle with radius 3. However, we cannot find the yellow shaded area without integrating the equation of the curve. This is why we need integration!

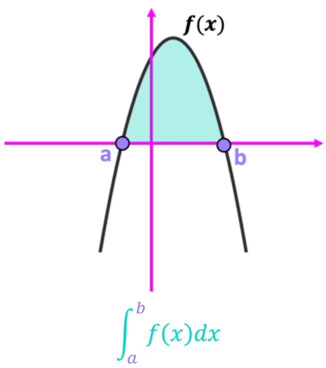
If you understand the section below and your integration techniques are already good, then this should be an easy topic!

2 How To Solve All Types Of Questions

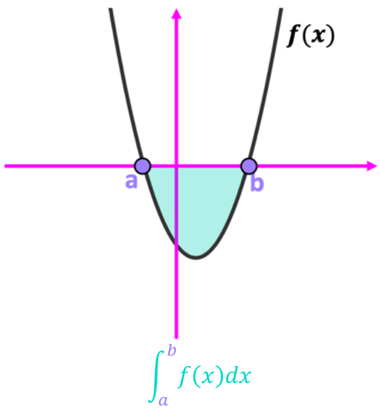
2.1 One Curve – One Shaded Area

As already mentioned, integrating a curve equation gives us the area between the **curve and the x axis** between the points on the x axis a and b.

$$\int_a^b (\text{curve equation})\, dx$$



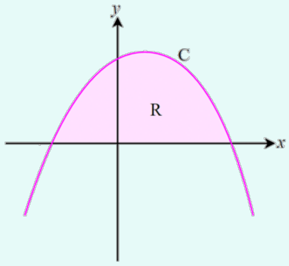
What about if the curve is **under** the x axis?



We will get a negative answer which just tells us it is underneath. We take the positive version as our area answer.

Consider a curve with equation $y = 4 + 3x - x^2$. Find the exact value of the area bounded by the curve and the x axis.

Let's graph this since we aren't given the graph



In order to find the x intercepts (and hence the integral limits) we set $y = 0$

$$4 + 3x - x^2 = 0$$
$$(1 + x)(4 - x) = 0$$
$$x = -1, x = 4$$

These are our limits for our integral

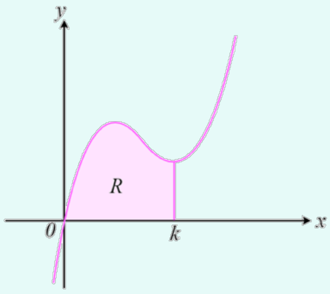
$$\int_{-1}^{4} (4 + 3x - x^2) dx$$

$$= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^{4}$$

$$= \left(4(4) + \frac{3(4)^2}{2} - \frac{(4)^3}{3}\right) - \left(4(-1) + \frac{3(-1)^2}{2} - \frac{(-1)^3}{3}\right)$$

$$= \frac{56}{3} - -\frac{13}{6} = \frac{125}{6} = 20\frac{5}{6}$$

The diagram below shows a sketch of part of the curve with equation $y = 2x^3 - 17x^2 + 40x$



The curve has a minimum turning point at $x = k$. The region R, is bounded by the curve, the x axis and the line with equation $x = k$. Show that the area of R is $\frac{256}{3}$

$$y = 2x^3 - 17x^2 + 40x$$
$$\frac{dy}{dx} = 6x^2 - 34x + 40$$

we need the value of k for our upper limit of the integral. This is a min point and hence where $\frac{dy}{dx} = 0$

(see differentation if you struggle with this)

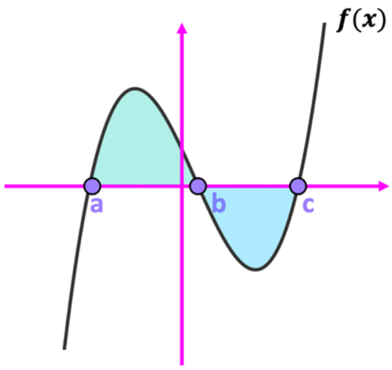
When $\frac{dy}{dx} = 0$,

$$6x^2 - 34x + 40 = 0$$
$$3x^2 - 17x + 20 = 0$$
$$(x - 4)(3x - 5) = 0$$
$$x = 4, x = \frac{5}{3}$$
$$\therefore k = 4$$

Area= $\int_0^4 (2x^3 - 17x^2 + 40x)\,dx = \left[\frac{x^4}{2} - \frac{17x^3}{3} + 20x^2\right]_0^4 = \frac{4^4}{2} - \frac{17(4)^3}{3} + 20(4)^2 - 0 = \frac{256}{3}$

2.2 One Curve – Two Or More Shaded Areas

When we have more than one shaded region, we have to deal with region separately and hence we will have two separate integrals.



$$\int_a^b f(x)dx + \int_b^c f(x)dx$$

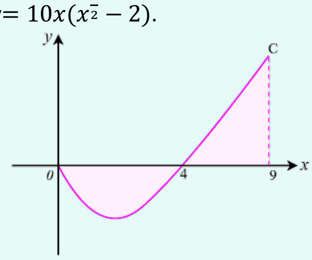
Note: $\int_b^c f(x)dx$ will be negative, so we need to use the positive version of this to get the total area

So, really we should write,

$$\int_a^b f(x)dx + \left|\int_b^c f(x)dx\right|$$

Where the | | means take the positive version

The diagram shows the curve C with equation y= $10x(x^{\frac{1}{2}} - 2)$.



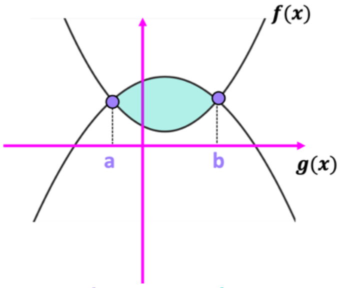
The curve starts at the origin and crosses the x axis at the point (4,0). The area shown shaded consists of two finite regions and is bounded by the curve C, the x axis and the line x=9. Find the total area of the shaded region

$$\int_0^4 10x\left(x^{\frac{1}{2}} - 2\right)dx + \int_4^9 10x\left(x^{\frac{1}{2}} - 2\right)dx$$
$$= +32 + 194$$
$$= 226$$

Note: We use 32 and not -32 since the area must be positive

2.3 Two Curves – Finding The Area BETWEEN Two Curves

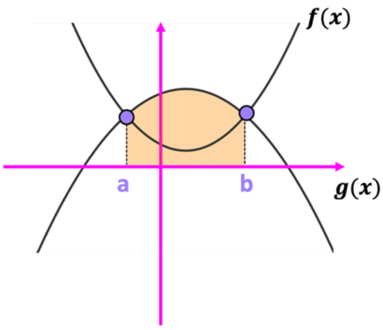
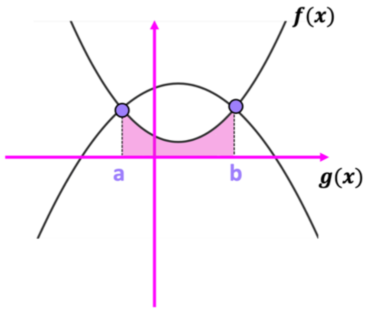
We are no longer finding the area between the curve and the x axis. Instead we are finding the area between two curves (or a line and a curve).



$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b (g(x) - f(x))dx$$

Notice how the limits are now the intersection points, not the values on the x axis.

Why is this formula true?

$\int_a^b g(x)dx$ gives the orange area	$\int_a^b f(x)dx$ gives the pink area
	

If we subtract the pink area from the orange area we get the green area

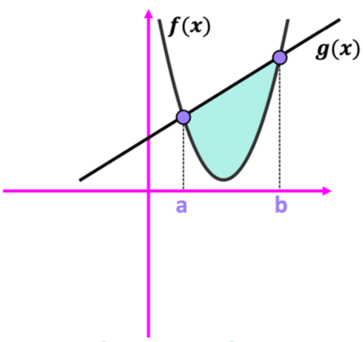
$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b (g(x) - f(x))dx$$

So you can remember between two curves as

$$\int_a^b (\text{top curve} - \text{bottom curve})dx$$

Remember the simplify top curve − bottom curve if possible first before integrating

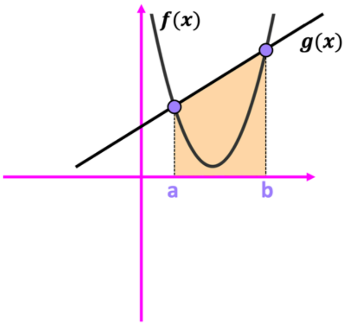
What about if we have a line and a curve such as:



$$\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b (g(x) - f(x))dx$$

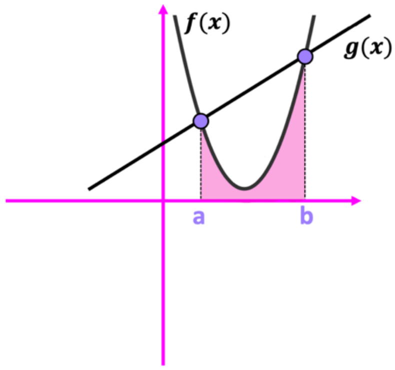
Why is this formula true?

$\int_a^b g(x)dx$ gives the orange area.



Note: We could also do the area of a trapezium instead of integrating this since we have a trapezium and know the area of it

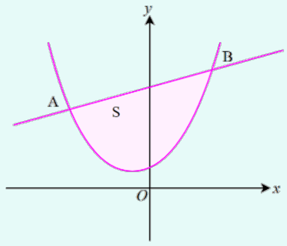
$\int_a^b f(x)dx$ gives the pink area



If we subtract the pink area from the orange area we get the green area

$\int_a^b g(x)dx - \int_a^b f(x)dx$ or area of trapezium $- \int_a^b f(x)dx = \int_a^b (g(x) - f(x))dx$

The line with equation $y = 3x + 20$ cuts the curve with equation $y = x^2 + 6x + 10$ at the points A and B



The shaded finite region is bounded by the line and the curve and is shown in the diagram.
Use calculus to find the exact shaded area

We need the intersection points first $3x + 20 = x^2 + 6x + 10$
$x^2 + 3x - 10 = 0$
$(x + 5)(x - 2) = 0$
$x = -5, x = 2$

Way 1:

Area$=\int_a^b (top\ curve)dx - \int_a^b (bottom\ curve)dx$

$\int_{-5}^2 (3x + 20)dx - \int_{-5}^2 (x^2 + 6x + 10)dx$

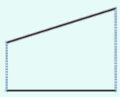
This can just be combined as one integral

$$\int_{-5}^2 ((3x + 20) - (x^2 + 6x + 10))dx$$

Simplify the integral
$\int_{-5}^2 (3x + 20 - x^2 - 6x - 10)dx$
$\int_{-5}^2 (-x^2 - 3x + 10)dx = \frac{343}{6} = 57\frac{1}{6}$

Way 2:

We can also do it as area of trapezium – area of curve

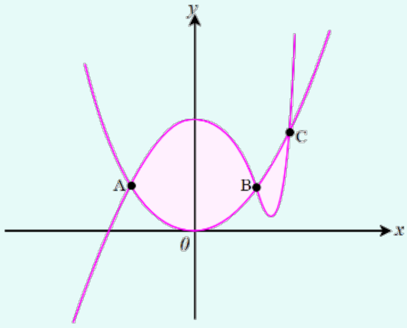
= area of  $- \int_a^b curve\ dx$

This should make sense since it is saying the whole area – (area under bottom curve) which gives us the pink shaded area.
So here, instead of integrating the line we can just use the fact that we know the area of a trapezium instead and not need to integrate the line $f(x)$

Area of trapezium
We need the y coordinates first for the points A and B
$x = 2: y = 3(2) + 20 = 26$ or $y = (2)^2 + 6(2) + 10 = 26$
$x = -5: y = 3(-5) + 20 = 5$ or $y = (-5)^2 + 6(-5) + 10 = 5$
Area $= \frac{1}{2}(26+5)(7) = \frac{217}{2}$

	$\frac{217}{2} - \int_{-5}^{2}(x^2+6x+10)dx = \frac{217}{2} - \frac{154}{3} = \frac{343}{6} = 57\frac{1}{6}$

The sketch below shows the curves $y = x^2$ and $y = x^3 - 4x + 4$



i. Find the coordinates of points of intersection of A, B and C

ii. Find the shaded region trapped between the 2 curves

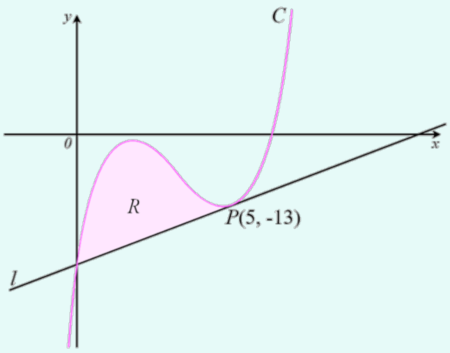
$$x^2 = x^3 - 4x + 4$$
$$x^3 - x^2 - 4x + 4 = 0$$
$$(x-2)(x+2)(x-1) = 0$$
$$x = -2, 1, 2$$

Need to substitute these into $y = x^2$ or $y = x^3 - 4x + 4$

$$x = -2: y = (-2)^2 = 4$$
$$x = 1: y = (1)^2 = 1$$
$$x = 2: y = (2)^2 = 4$$
$$(-2,4), (1,1), (2,4)$$

$$\int_{-2}^{1}[(x^3 - 4x + 4) - x^2]dx + \int_{1}^{2}[x^2 - (x^3 - 4x + 4)]dx$$
$$\int_{-2}^{1}(x^3 - x^2 - 4x + 4)dx + \int_{1}^{2}(-x^3 + x^2 + 4x - 4)dx$$

$$= \frac{45}{4} + \frac{7}{12} = \frac{71}{6}$$



The diagram shows a part of the curve C with equation $y = x^3 - 10x^2 + 27x - 23$

The point P$(5, -13)$ lies on C. The line l is tangent to C at P

i. verify that l meets C again on the y axis

The region R, shown shaded, is bounded by the curve C and the line l

ii. Use algebraic integration to find the exact area of R

$$\frac{dy}{dx} = 3x^2 - 20x + 27$$

When $x = 5$:

$\frac{dy}{dx} = 3x = (5)^2 - 20(5) + 27 = 2$

We can find the equation of the tangent passing through the point $(5,13)$ with gradient 2

$y - 13 = 2(x - 5)$

$y = 2x - 23$

When $x = 0$: $y = (0)^3 - 10(0)^2 + 27(0) - 23 = -23$

$(0, -23)$ also passes through the line $y = 2x - 23$ so C meets l again on the y axis

$$\int_0^5 ((x^3 - 10x^2 + 27x - 23) - (2x - 23))dx$$

$$\int_0^5 (x^3 - 10x^2 + 25x)dx$$

$$\left[\frac{x^4}{4} - \frac{10x^3}{3} + \frac{25x^2}{2}\right]_0^5$$

$$\left(\frac{5^4}{4} - \frac{10(5)^3}{3} + \frac{25(5)^2}{2}\right) - (0)$$

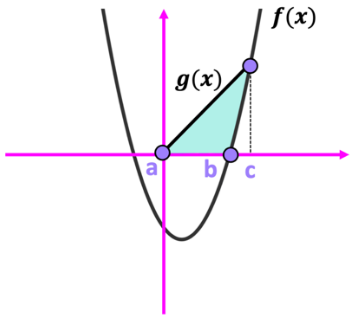
$$\frac{625}{12}$$

2.4 Two Curves – Finding An Area Which Is NOT Between

We can either have :

1. an area that is not completely contained between two curves (or a line and a curve). For this we find the entire area with the x axis and subtract the part that we don't need
2. an area that is formed by two different curves (or by a different line and curve). For this we find two separate areas and add them together

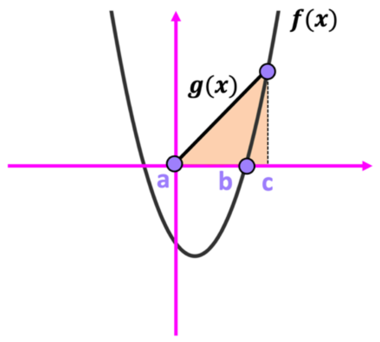
Let's look at an example of 1. first



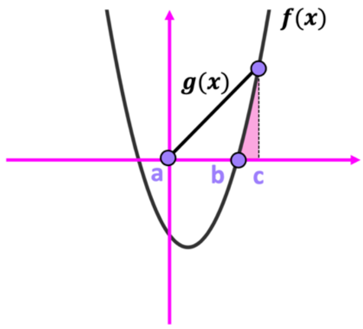
$$\int_a^c g(x)dx - \int_b^c f(x)dx$$

Why is this formula true?

$\int_a^b g(x)dx$ gives the orange area (entire area)	$\int_b^c f(x)dx$ gives the pink area (part we don't't need)



Note: We could also do the area of a triangle instead of integrating this since we have a triangle and know the area of it



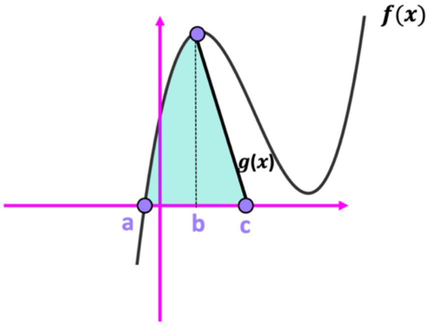
If we subtract the pink area from the orange area we get the green area

$$\int_a^c g(x)dx - \int_b^c f(x)dx = \int_a^c g(x)dx - \int_b^c f(x)dx$$

Notice that the line forms a triangle, so we could have instead done:

area of triangle $- \int_b^c f(x)dx$

Now let's look at an example for 2. The area from a to b is given by the straight line and the area from b to c is given by the curve.

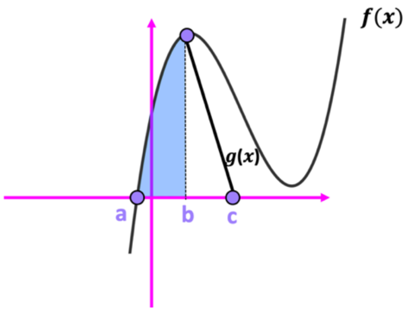


$$\int_a^b f(x)dx + \int_b^c g(x)dx$$

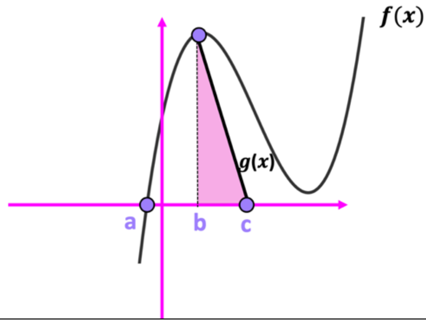
Take notice of how you need the x coordinate of maximum point for limit b.

Why is this formula true?

$\int_a^b f(x)dx$ gives the blue area



$\int_b^c f(x)dx$ gives the pink area



	Note: We could also do the area of a triangle instead of integrating this since we have a triangle and know the area of it

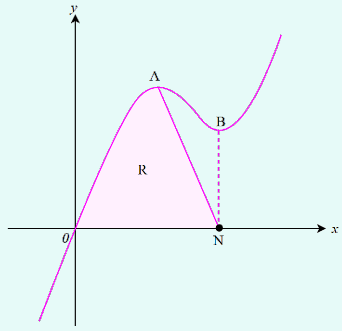
If we add the pink area to the blue area we get the green area

$$\int_a^b f(x)dx + \int_b^c g(x)dx = \int_a^b f(x)dx + \int_b^c g(x)dx$$

Notice that the line forms a triangle, so we could have instead done:

$$\int_a^b f(x)dx + \text{area of triangle}$$

The diagram above shows a sketch of part of the curve with equation $y = x^3 - 8x^2 + 20x$. The curve has stationary points at A and B. The line through B parallel to the y axis meets the x axis at the point N. The region R shown is bounded by the curve, the x axis and the line from A to N.



Calculate the exact area of R

$$y = x^3 - 8x^2 + 20x$$
$$\frac{dy}{dx} = 3x^2 - 16x + 20$$
$$3x^2 - 16x + 20 = 0$$
$$x = \frac{10}{3}, 2$$

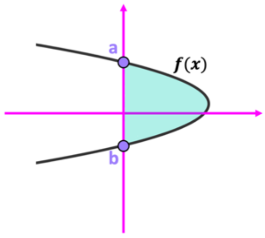
$A(2,16)$. We need this coordinate since we need y for the height of the triangle later.

$$\int_0^2 (x^3 - 8x^2 + 20x)dx + \text{area of triangle}$$
$$\int_0^2 (x^3 - 8x^2 + 20x)dx + \frac{1}{2}\left(\frac{10}{3} - 2\right)(16)$$
$$\frac{68}{3} + \frac{32}{3} = \frac{100}{3}$$

2.5 About The Y Axis

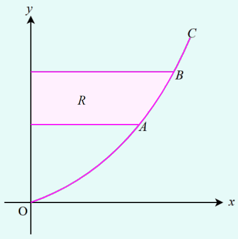
Careful when finding the area about the y axis (so far we have just found the area about the x axis). We are normally given the function as y in terms of x. We now we need to do 2 things differently.

- re-arrange to get x in terms for y for $f(y)$ part
- use y limits not x. This means if you're not given the y limits and x instead, you will need to plug x limits in to find what y should be



$$\int_a^b f(y)dy$$

The curve C, as shown, represents the graph of $y = \frac{x^2}{25}$.



The points A and B on the curve C have x coordinates 5 and 10 respectively. The finite region R is enclosed by C, the y axis and the lines through A and B parallel to the x axis. Use integration to find the area of R.

We need the function in terms of y instead with y limits since we want the area about the y axis

Firstly, we change the limits to be y:

$x = 5: y = \frac{25}{25} = 1$

$x = 10: y = \frac{100}{25} = 4$

Now let's re-arrange the function to be in the form $y = \cdots$:

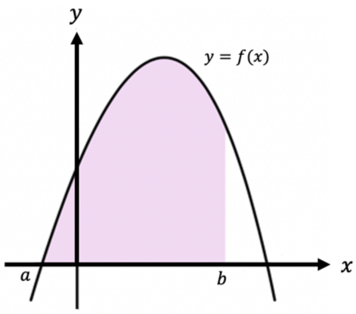
$y = \frac{x^2}{25} \Rightarrow x^2 = \sqrt{25y} \Rightarrow x = 5y^{\frac{1}{2}}$

So we get

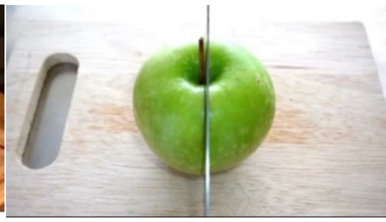
$$\int_1^4 5y^{\frac{1}{2}}\,dy = \frac{70}{3}$$

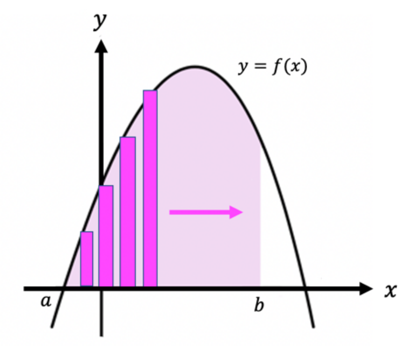
3 Why Integration Gives The Area

Let's say we wanted to find the pink area under the curve below.

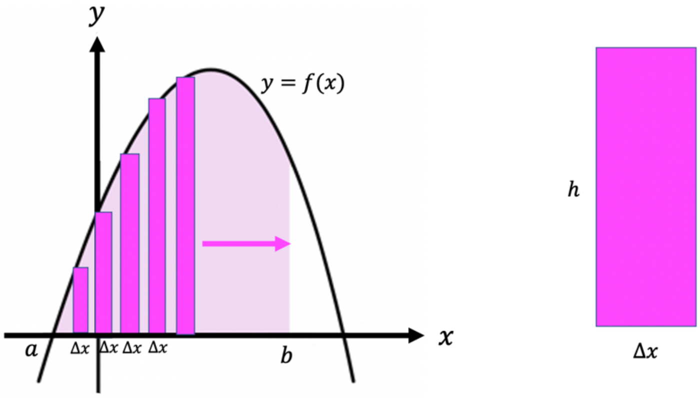


This is not a familiar shape that we know the area of, but we can turn it into lots of familiar shapes. We do this by cutting lots of vertical slices perpendicular to the x axis (like cutting a watermelon or an apple)

These slices create lots of rectangles. Each rectangle is very thin and hence we say has width Δx (Δx can open be seen as δx instead, so don't get confused if your teachers uses a different notation). The two notations are used in maths to denote something very tiny). So, each strip is a rectangle with thickness Δx.

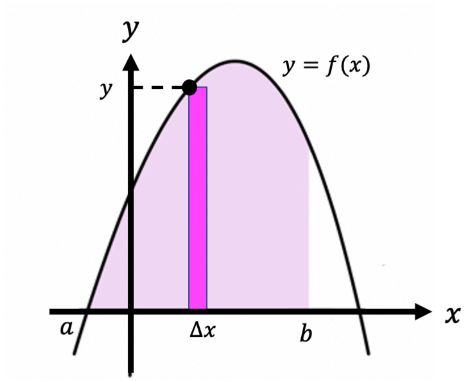


Rectangles have an area of base times height
This means that we can say that:

$$\Delta A = (\text{Base})(\text{height}) = \Delta x \times h = \Delta x\, h$$

where ΔA is small change in amount of area (we use the delta notation since the strip makes a small contribution to the total area)

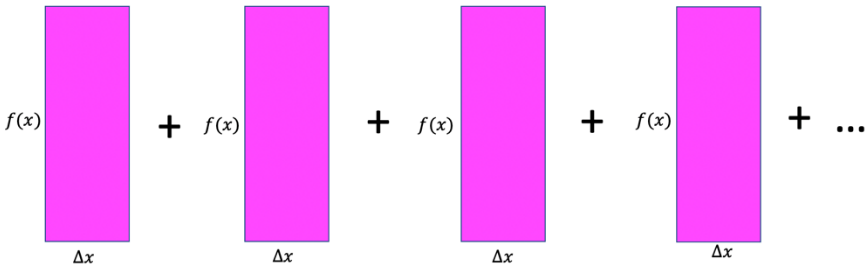
The equation of the curve gives the height function, so we can replace the height with $f(x)$ or y



So instead of writing $\Delta A = \Delta x\, h$, we can write

$$\Delta A = \Delta x\, f(x)$$

The sum of the areas of all these rectangles gives the total area of the shape, so to get the area of the shape, we want to add up the area of all the rectangles between the x coordinates of a and b.



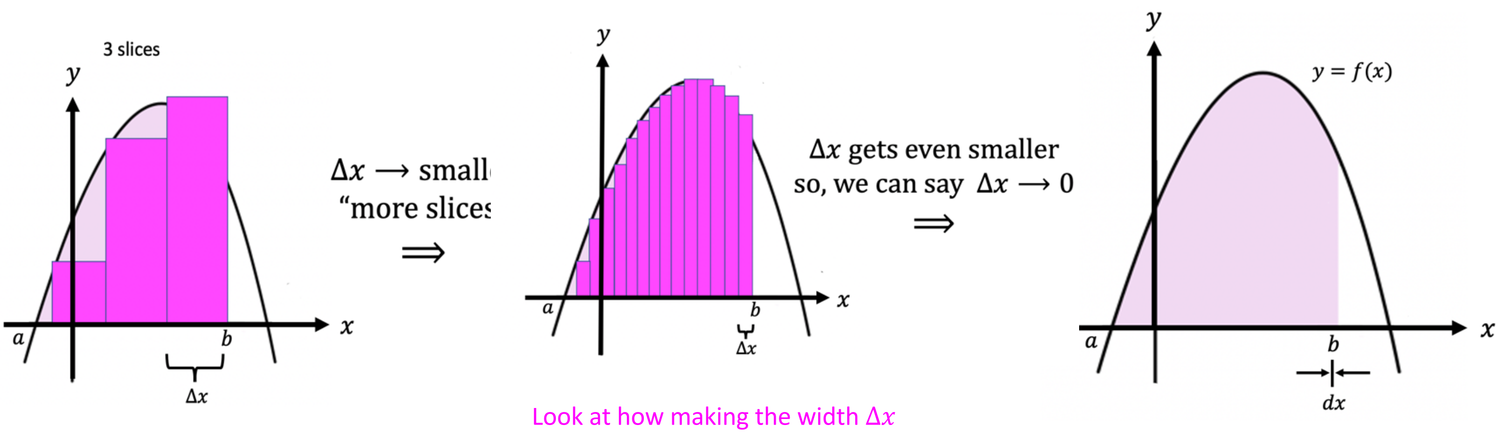
You might be thinking that this is a lot of adding, but thankfully there is a shortcut! Somehow an integral appears and saves us. Let's see how.

we have already shown that $\Delta A = f(x)\Delta x$

Total area = sum of areas of all the strips
We use the symbol $\sum$ to mean sum
So, $\sum_{x=a}^{b} \Delta A \approx \sum_{x=a}^{b} f(x)\Delta x$

To make the approximation more accurate we must let the thickness of each strip become very small indeed, that is we let $\Delta x \to 0$. The smaller we make the widths of these strips, δx, the more rectangles we can fit in and hence the more accurate the area will be. We write this as

Let's see this with a picture before we discuss it more formally:



We could add the areas of these rectangles to find the area, but as we can see, we will not get a very accurate answer

Look at how making the width Δx smaller gives a more accurate answer

The widths of the slices approach zero and the answer now approaches the true answer. Notice how we write dx instead of Δx now to mean the slices are approaching zero width (we can no longer even see any rectangles as they as so tiny

To denote that the thickness of the strips has become very small we write

$$\lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\Delta x$$

Why did we apply the limit to the sum? The notation $\lim_{\Delta x \to 0}$ means that we consider what happens to the expression following it as $\Delta x \to 0$ gets smaller and smaller. This is known as the limit of a sum. If the limit exists we write this formally as an integral:

$$\int_a^b f(x)\,dx$$

Thus, we have defined an integral as the limit of a sum, so

$$\lim_{\Delta x \to 0} \sum_{x=a}^{b} f(x)\Delta x = \int_a^b f(x)\,dx$$

Don't get too caught up with the confusing notation above. All you really need to remember to be good at this topic is the formula $\int_a^b f(x)\,dx$ and that to find area under a curve for function $f(x)$, integrate the function between the correct limits!

Struggling to remember the formula? It may help to have this in mind:

$$\int_a^b dA = Area\ of\ rectanges = \int_a^b (height\ of\ rectangle)\,(width\ of\ rectangle) = \int_a^b h\,dx = \int_a^b f(x)\,dx$$

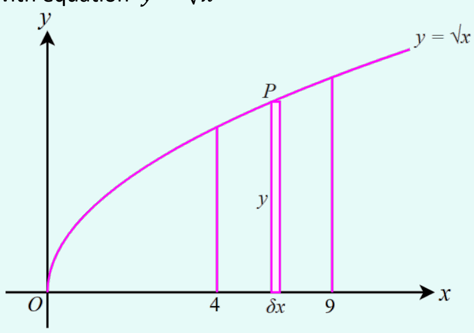
Note: we can also find the area the y axis and our formula becomes

$$\int_a^b f(y)\,dx$$

where a and b are the y limits now instead of x and $f(y)$ is the original equation rearranged for x in terms of y for

See Riemann Sums worksheet for more detail on this!

The diagram shows a sketch of the curve with equation $y = \sqrt{x}$



The point P(x,y) lies on the curve

The rectangle shown shaded has height y and width δx

Calculate $\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\ \delta x$

This just means $\int_4^9 \sqrt{x}dx$

$$\lim_{\delta x \to 0} \sum_{x=4}^{9} \sqrt{x}\,\delta x = \int_4^9 \sqrt{x}\,dx = \int_4^9 x^{\frac{1}{2}}\,dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^9 = \left[\frac{2x^{\frac{3}{2}}}{3}\right]_4^9 = \frac{2}{3}(9)^{\frac{3}{2}} - \frac{2}{3}(4)^{\frac{3}{2}} = \frac{38}{3}$$

4 Summaries

If we just had straight lines we could use our familiar formulae for triangles, rectangles, trapezium etc to find areas, but now we have curves and we don't know the areas of the shapes, so we need integration to help us! If we have familiar shapes and know a formula for their area we would not bother to integrate! Integrating horizontal lines gives us the area of familiar rectangles or squares and diagonal lines gives us area of familiar triangles or trapezia. But we don't have this luxury anymore with curves!

4.1 About The X Axis

Integration on a curve finds you the AREA between the **curve and the x axis**.

$\int curve\ equation$ = the area between the curve and the $x\ axis$

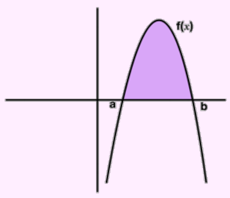
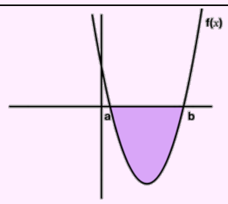
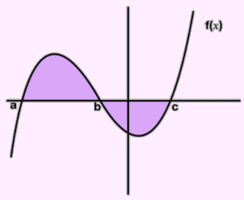
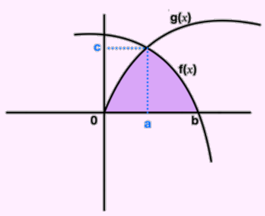
The area under a curve **between two points** can be found by doing a definite integral between the two points (i.e. by putting in limits we can specify the certain region that we want). So, to find the area under the curve $y = f(x)$ between $x = a$ and $x = b$, we integrate $y = f(x)$ between the limits of a and b.

$$\text{Area}=\int_{x=a}^{x=b} f(x)dx = \int_{x=a}^{x=b} y\ dx = \int_{x=a}^{x=b} (y\ in\ terms\ of\ x)\ dx$$

Areas under the x-axis will come out negative and areas above the x-axis will be positive.

Note: You are not always given the limits or curve/line equations: you might need to find the x intercepts or use geometry/calculus to get line or curve equation

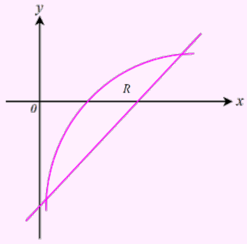
4.1.1 Between Curve And X Axis

 Area=$\int_a^b f(x)dx$ Why? Really we are doing Area=$\int_a^b top\ dx - \int_a^b bottom\ dx$ $= \int_a^b f(x)\ dx - \int_a^b 0\ dx$ $= \int_a^b f(x)\ dx$	 Area=$-\int_a^b f(x)dx$ Why? Really we are doing Area=$\int_a^b top\ dx - \int_a^b bottom\ dx$ $= \int_a^b 0\ dx - \int_a^b f(x)\ dx$ $= -\int_a^b f(x)\ dx$ Note: If just integrating generally we wouldn't put the $-$, but we want areas to be positive, so we put the negative to make the integral positive, since integrals under the x axis are negative OR we can just do $\int_a^b f(x)dx$ and take the positive version of our negative answer An integral for underneath an axis gives a negative answer. If want area give positive answer, if want value of integral only give negative answer  	 We have to do 2 separate integrals here. This question is a combination of the 2 types on the left Area=$\int_a^b f(x)dx + -\int_b^c f(x)dx$ Note: If just integrating we wouldn't put the $-$, but finding areas we want both to be positive so we put the negative to make the negative integral positive. Alternatively we can just find both integrals and take the positive versions of any negative answers and add our answers together Similarly for 2 different curves, we must do the integrals separately with the different curves  Area=$\int_0^a g(x)dx + \int_a^b f(x)dx$ (notice how we needed the intersection point a)

4.1.2 Between Two Curves

Think every time you integrate it gives you the area between that line/curve and the axis. So when finding area between two curves you first find the area under the top curve/line between the curve and the axis and then you need to take away the area under the bottom part between the bottom line/curve and axis to get the enclosed part

The area is completely contained between the two curves. It doesn't have to be "one directly on top of the other, like

as 

There is a slight slant, but the area is still contained between the curves. Need to find intersections for the limits now.

The fastest way to do this is

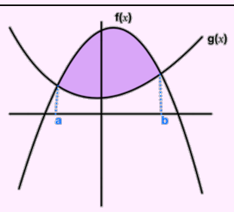
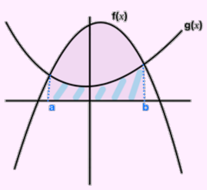
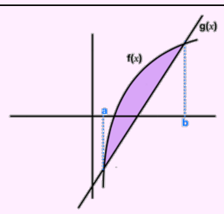
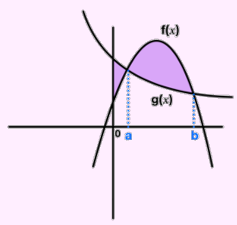
$$\int top\ curve - \int bottom\ curve$$

This can be combined as one integral

$$\int (top\ curve - bottom\ curve)$$

See ***IMPORTANT*** below for the alternate way of dealing with these

Here we need to find the **intersection points** of the curves first since our limits are no longer the x intercepts!
Note: you might need to solve simultaneously to get intersections if not given or use geometry/calculus to get line or curve equation if not given that

 Area=$\int_a^b f(x)dx - \int_a^b g(x)dx$ This can be combined as 1 integral $\int_a^b [f(x) - g(x)]dx$ Why do we do $\int top\ curve - \int bottom\ curve$? $\int_a^b f(x)dx$ finds the area between the curve $f(x)$ and the x axis which is all of the pink shaded region below $\int_a^b g(x)dx$ finds the blue shaded region below. Subtracting these answers will give us the purple region above that we want.  	 This often confuses students for some reason. It is exactly the same as the example on the left. The area is still contained between 2 curves and it is clear which is the top and bottom curve which doesn't change the whole time $\int_a^b [f(x) - g(x)]dx$	 This is the same except we have to do 2 separate integrals since the top curve changes in both regions $\int_0^a [g(x) - f(x)]dx$ $+\int_a^b [f(x) - g(x)]dx$

IMPORTANT
Any time straight lines are involved we can use a combination of known areas of shapes such as triangles, rectangles and trapezia to save time when finding areas. As mentioned earlier, if we have familiar shapes and know a formula for their area why bother to integrate! Integrating horizontal lines gives us the area of rectangles or squares and diagonal lines gives us area of triangles or trapezia.
e.g. 1

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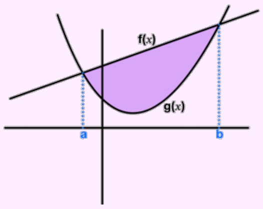
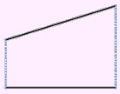
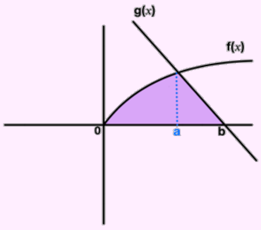
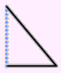


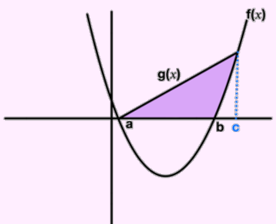
Way 1:	Way 2:
This can be done in the same way as the types above: Area= $\int_a^b (top\ curve)dx - \int_a^b (bottom\ curve)dx$ $\int_a^b f(x)dx - \int_a^b g(x)dx$ This can just be combined as one integral $\int_a^b (f(x) - g(x))dx$	We can also do it as area of trapezium – area of curve = area of  $-\int_a^b g(x)dx$ This should make sense since it is saying the whole area – (area under bottom curve) which gives us the purple shaded area. So here, instead of integrating the line we can just use the fact that we know the area of a trapezium instead and not need to integrate the line $f(x)$

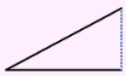
e.g. 2



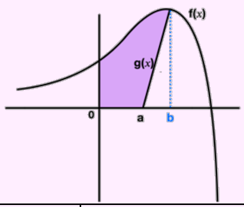
Way 1:	Way 1:
we do this as $\int_0^a f(x)dx + \int_a^b g(x)dx$ Take note of the limits and which curves they correspond to!	$\int_0^a f(x)dx +$ Area of  So here, instead of integrating the line we can just use the fact that we know the area of a triangle instead and not need to ACTUALLY integrate the line

e.g. 3
In this example doing top – bottom would not work as we want a region left out. We need to use triangles to help us or integrate the line instead. This example is good as it really tests if you understand which regions integrating actually finds us.



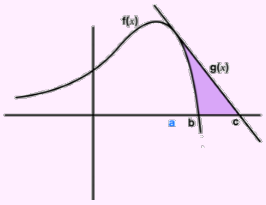
Way 1:	Way 2:
we do this as $\int_a^c g(x)dx - \int_b^c f(x)dx$ Take note of the limits and which curves they correspond to!	area of  $-\int_b^c f(x)dx$ So here, instead of integrating the line we can just use the fact that we know the area of a triangle instead and not need to actually integrate the line

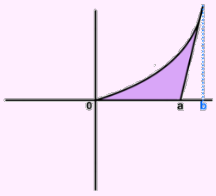
e.g. 4
In this example doing top – bottom would not work as we want a region left out. We need to use triangles to help us or integrate the line.

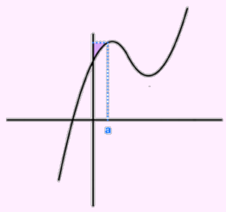


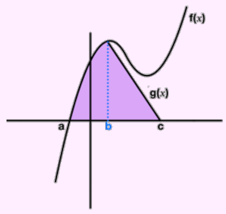
Way 1:	Way 2:
we do this as $\int_0^b f(x)dx - \int_a^b g(x)dx$ Take note of the limits and which curves they correspond to!	 area of $\int_0^b f(x)dx$ – Area of

Have a go at the following 4 examples. You should now be able to set up the integrals for all of them if you've understood the above well







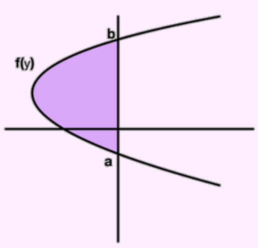
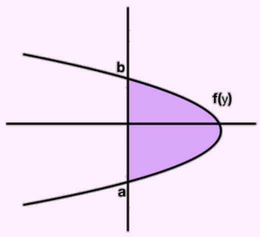
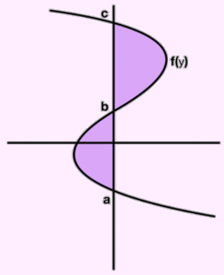


4.2 About The Y Axis

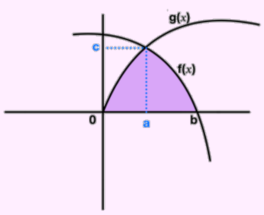
What happens when you want the area between the curve and the **y axis instead**? We don't always have to use the x axis
The only difference is that we now have y limits instead and the function is in terms of y. Think of turning your page around so the y axis looks the same as the x axis did

So, to find the area under the curve x= $f(y)$ between y= a and y= b, we integrate x= $f(y)$ between the limits of a and b.

$\int_{y=\cdots}^{y=\cdots} f(y)dy = \int_{y=\cdots}^{y=\cdots} x\, dy$

 $-\int_a^b f(y)dy$ This is negative for the same reason that under x axis means negative	 $\int_a^b f(y)dy$	 $-\int_a^b f(y)dy + \int_b^c f(y)dy$

If 2 separate regions:



Area about x axis=$\int_{x=0}^{x=a} g(x)dx + \int_{x=a}^{x=b} f(x)dx$
Area about y axis=$\int_{y=0}^{y=c} [f(y) - g(y)]dy$

4.3 Formulae

4.3.1 About The X Axis

Find the area between the curve $f(x) = x^2 - 4$ and the x axis

Step 1: Plug in y in terms of x

Our template is $\int_{x=\dots}^{x=\dots} (y)dx$, where dx just tells us this is about x axis

$\int_{x=\dots}^{x=\dots} (x^2 - 4)dx$

Step 2: Plug in x limits or find them if not given

$x^2 - 4 = 0 \Longrightarrow x = \pm 2$

$\int_{x=-2}^{x=2} (x^2 - 4)dx$

Step 3: Integrate as normal

$\left[\frac{x^3}{3} - 4x\right]_{-2}^{2}$ etc

4.3.2 About The Y Axis

Find the area between the curve $f(x) = 2x + 1$ and the lines $y = 1, y = 3$

Step 1: Plug in x in terms of y (might need to rearrange to get x in terms of y first as you're usually given y in terms of x)

Our template is $\int_{y=\dots}^{y=\dots} (x)dy$, where dy just tells us this is about y axis

We need to re-arrange to get x in terms of y

$y = 2x + 1 \Leftrightarrow x = \frac{y-1}{2}$

$\int_{y=\dots}^{y=\dots} (\frac{y-1}{2})dy$

Step 2: Plug in y limits or find them if not given

$\int_{y=1}^{y=3} (\frac{y-1}{2})dy$

Step 3: Integrate as normal

$\left[\frac{y^2}{4} - \frac{1}{2}y\right]_{1}^{3}$ etc