

Ch. IX-B, PART C

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1. a) $(1+x)^{\frac{1}{2}} = 1 + \frac{1}{2}(x) + \frac{\frac{1}{2}(-\frac{1}{2})}{1 \times 2} x^2 + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1 \times 2 \times 3} x^3 + o(x^4)$
 $= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + o(x^4)$

b) $\sqrt{4+2x} = (4+2x)^{\frac{1}{2}} = 4^{\frac{1}{2}} (1 + \frac{1}{2}x)^{\frac{1}{2}} = 2(1 + \frac{1}{2}x)^{\frac{1}{2}}$
 $= 2 \left[1 + \frac{1}{2}(\frac{1}{2}x) - \frac{1}{8}(\frac{1}{2}x)^2 + \frac{1}{16}(\frac{1}{2}x)^3 + o(x^4) \right]$
 $= 2 \left[1 + \frac{1}{4}x - \frac{1}{32}x^2 + \frac{1}{128}x^3 + o(x^4) \right]$
 $= 2 + \frac{1}{2}x - \frac{1}{16}x^2 + \frac{1}{64}x^3 + o(x^4)$

c) VALID FOR $|\frac{1}{2}x| < 1$
 $|x| < 2 \quad \therefore -2 < x < 2$

2. $x^3 + 2xy = e^y$
 $\Rightarrow \frac{d}{dx}(x^3) + \frac{d}{dx}(2xy) = \frac{d}{dx}(e^y)$
 $\Rightarrow 3x^2 + 2y + 2x \frac{dy}{dx} = e^y \frac{dy}{dx}$
 $\Rightarrow 3x^2 + 2y = (e^y - 2x) \frac{dy}{dx}$

$\Rightarrow \frac{3x^2 + 2y}{e^y - 2x} = \frac{dy}{dx}$

BUT $e^y = x^3 + 2xy$

$\therefore \frac{dy}{dx} = \frac{3x^2 + 2y}{(x^3 + 2xy) - 2x}$

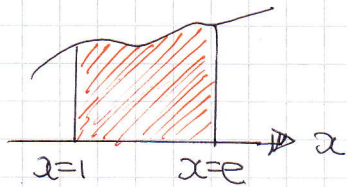
$\frac{dy}{dx} = \frac{3x^2 + 2y}{x^3 + 2xy - 2x}$

As Required

Q4, YGB, PAPER C

-2-

3.



$$V = \pi \int_{x_1}^{x_2} (y(x))^2 dx$$

IN THIS CASE

$$V = \pi \int_1^e [x^{\frac{3}{2}} \sqrt{\ln x}]^2 dx$$

Thus

$$V = \pi \int_1^e x^3 \ln x dx = \dots \text{BY PARTS AND IGNORING THE UNITS AND } \pi \text{ FOR THE TIME BEING } \dots$$

$\ln x$	$\frac{1}{x}$
$\frac{1}{4}x^4$	x^3

$$\dots = \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^4 \times \frac{1}{x} dx$$

$$= \frac{1}{4}x^4 \ln x - \int \frac{1}{4}x^3 dx$$

$$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + C$$

REINTRODUCING π & UNITS

$$= \pi \left[\frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 \right]_1^e$$

$$= \pi \left[\left(\frac{1}{4}e^4 \ln e - \frac{1}{16}e^4 \right) - \left(\frac{1}{4} \times 1^4 \ln 1 - \frac{1}{16} \right) \right]$$

$$= \pi \left[\frac{1}{4}e^4 - \frac{1}{16}e^4 + \frac{1}{16} \right]$$

$$= \pi \left[\frac{3}{16}e^4 + \frac{1}{16} \right]$$

$$= \frac{1}{16}\pi (3e^4 + 1)$$

REVIEW

CH 17GB, PAPER C

4. a)

• when $x=0$

$$0 = t^2 - 8t + 12$$

$$0 = (t-2)(t-6)$$

$$t = \begin{cases} 2 \\ 6 \end{cases} \quad y = \begin{cases} -2 \\ 2 \end{cases}$$

$$\therefore (0, -2) \quad (0, 2)$$

• when $y=0$

$$0 = t - 4$$

$$t = 4$$

$$x = 4^2 - 8 \times 4 + 12$$

$$x = 16 - 32 + 12$$

$$x = -4$$

$$\therefore (-4, 0)$$

b)

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1}{2t-8}$$

$$\left. \frac{dy}{dx} \right|_{t=5} = \frac{1}{2 \times 5 - 8} = \frac{1}{2}$$

NORMAL GRADIENT = -2

$$y - y_0 = m(x - x_0)$$

$$y - 1 = -2(x + 3)$$

$$y - 1 = -2x - 6$$

$$y + 2x + 5 = 0$$

AS REQUIRED

AT $P(-3, 1)$



$t=5$
BY INSPECTION OF
 $y = t - 4$

c)

$$y = t - 4$$

$$t = y + 4$$

SUB INTO THE OTHER

$$x = (y+4)^2 - 8(y+4) + 12$$

$$x = y^2 + 8y + 16 - 8y - 32 + 12$$

$$x = y^2 - 4$$

$$y^2 = x + 4$$

C4, MGB, PAPER C

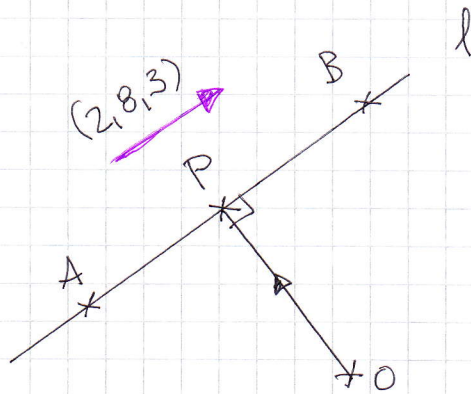
5. a) $\vec{AB} = \underline{b} - \underline{a} = (13, 23, 7) - (11, 15, 4) = (2, 8, 3)$

$\underline{r} = \text{fixed point} + \lambda (\text{direction})$

$\underline{r} = (11, 15, 4) + \lambda (2, 8, 3)$

$\underline{r} = (2\lambda + 11, 8\lambda + 15, 3\lambda + 4)$

b)



● LET $P(x, y, z)$

OR $P = (x, y, z)$

● $\vec{OP} \perp l$

$(x, y, z) \cdot (2, 8, 3) = 0$

$2x + 8y + 3z = 0$

● P lies on l

$x = 2\lambda + 11$

$y = 8\lambda + 15$

$z = 3\lambda + 4$

THUS $2(2\lambda + 11) + 8(8\lambda + 15) + 3(3\lambda + 4) = 0$

$4\lambda + 22 + 64\lambda + 120 + 9\lambda + 12 = 0$

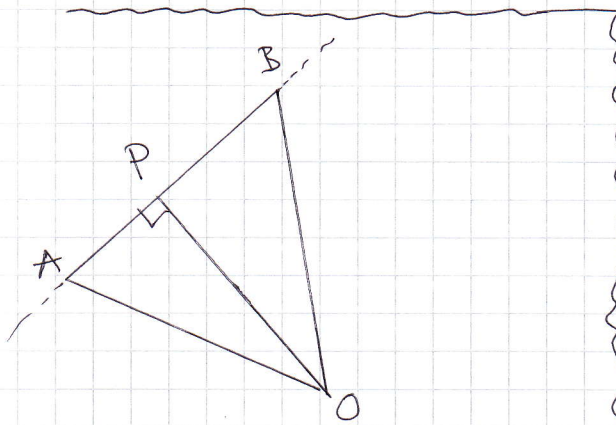
$77\lambda = -154$

$\lambda = -2$

∴ USING $\underline{r} = (2\lambda + 11, 8\lambda + 15, 3\lambda + 4)$

GIVES $P(2(-2) + 11, 8(-2) + 15, 3(-2) + 4)$

$P(7, -1, -2)$
 ~~$P(7, -1, -2)$~~



● $|\vec{AB}| = |2, 8, 3| = \sqrt{4 + 64 + 9} = \sqrt{77}$

● $|\vec{OP}| = |7, -1, -2| = \sqrt{49 + 1 + 4} = \sqrt{54}$

$\text{Area} = \frac{1}{2} |\vec{AB}| |\vec{OP}|$

$= \frac{1}{2} \sqrt{77} \cdot \sqrt{54} = \frac{3}{2} \sqrt{462}$

≈ 32.24

6. a) "STEADY RATE" of 1.5 cm² PER SECOND ⇒

IN 4 SECONDS $A = 4 \times 1.5$

$$A = 6 \text{ cm}^2$$

$$\pi r^2 = 6$$

$$r^2 = \frac{6}{\pi}$$

$$r = 1.38 \text{ cm}$$

b) $\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$

$$\frac{dr}{dt} = \frac{1}{2\pi r} \times 1.5$$

$$\left. \frac{dr}{dt} \right|_{r=1.38} = \frac{1}{2\pi \times 1.38} \times 1.5$$

$$\left. \frac{dr}{dt} \right|_{t=4} \approx 0.173 \text{ cm s}^{-1}$$

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r$$

$$\frac{dr}{dA} = \frac{1}{2\pi r}$$

7. a) $\frac{dm}{dt} = -k(m-10)$

$$\Rightarrow dm = -k(m-10) dt$$

$$\Rightarrow \frac{1}{m-10} dm = -k dt$$

$$\Rightarrow \int \frac{1}{m-10} dm = \int -k dt$$

$$\Rightarrow \ln|m-10| = -kt + C$$

$$\Rightarrow m-10 = e^{-kt+C}$$

$$\Rightarrow m-10 = Ae^{-kt} \quad (A=e^C)$$

$$\Rightarrow m = 10 + Ae^{-kt}$$

b) $t=0 \quad m=120$

$$120 = 10 + Ae^0$$

$$120 = 10 + A$$

$$A = 110$$

$$m = 10 + 110e^{-kt}$$

• $60 = 10 + 110e^{-k \times 3}$

$$50 = 110e^{-3k}$$

$$\frac{5}{11} = e^{-3k}$$

$$\frac{11}{5} = e^{3k}$$

$$3k = \ln \frac{11}{5}$$

$$k = \frac{1}{3} \ln \frac{11}{5}$$

C4 NGB, PAPER C

- 6 -

c) $m = 10 + 110e^{-0.2628...t}$

$\frac{1}{3} \ln \frac{11}{5}$

with $t=6$

$m = 10 + 110e^{-0.2628... \times 6}$

$m \approx 32.7g = \frac{360}{11}$

8. a)

$\frac{2u^2}{(u-1)(u+1)} \equiv A + \frac{B}{u+1} + \frac{C}{u-1}$

$2u^2 \equiv A(u+1)(u-1) + B(u-1) + C(u+1)$

- If $u=1$, $2 = 2C \Rightarrow C=1$
- If $u=-1$, $2 = -2B \Rightarrow B=-1$
- If $u=0$, $0 = -A - B + C$
 $A = 1 - (-1)$
 $A = 2$

b)

$\int_3^8 \frac{\sqrt{x+1}}{x} dx = \dots$ SUBSTITUTION...

$= \int_2^3 \frac{u}{x} \cdot 2u du = \int_2^3 \frac{2u^2}{x} du$

$= \int_2^3 \frac{2u^2}{u^2-1} du = \int_2^3 \frac{2u^2}{(u-1)(u+1)} du$

$= \int_2^3 \left(2 + \frac{1}{u-1} - \frac{1}{u+1} \right) du$ (PART a)

$= \left[2u + \ln|u-1| - \ln|u+1| \right]_2^3$

$u^2 = x+1$
 $u = (x+1)^{\frac{1}{2}}$
 $2u \frac{du}{dx} = 1$
 $2u du = dx$
 $x=3, u=2$
 $x=8, u=3$
 $x = u^2 - 1$

C4, 1YGB, PAPER C -7-

$$= [6 + \ln 2 - \ln 4] - [4 + \cancel{\ln 4} - \ln 3]$$

$$= 2 + \ln 2 - \ln 4 + \ln 3$$

$$= \left(2 + \ln \frac{3}{2}\right)$$

g)

x	3	4	5	6	7	8
y	0.6667	0.5590	0.4899	0.4410	0.4041	0.3750

d)

$$\int_3^8 \frac{\sqrt{x+1}}{x} dx \approx \frac{\text{THICKNESS}}{2} [\text{FIRST} + \text{LAST} + 2 \times \text{RFST}]$$

$$\approx \frac{1}{2} [0.6667 + 0.3750 + 2(0.5590 + 0.4899 + 0.4410 + 0.4041)]$$

$$\approx 2.4479...$$

$$\approx 2.415$$

- e)
- EXACT VALUE = $2 + \ln \frac{3}{2} = 2 + \ln 2 - \ln 4 + \ln 3 \approx 2.405$
 - TRAPEZIUM RUL = 2.415

DIFFERENCE OF $2.415 - 2.405 = 0.01$

TRAPEZIUM OVERESTIMATES