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LYGB - FPI PAPER M - QUESTION 1

- $\underline{M} = \begin{pmatrix} k & k+1 \\ k+1 & k+2 \end{pmatrix}$
- $\det(\underline{M}) = k(k+2) - (k+1)(k+1) = k^2 + 2k - (k^2 + 2k + 1)$
 $= k^2 + 2k - k^2 - 2k - 1 = -1$
- $\underline{M}^{-1} = \frac{1}{-1} \begin{bmatrix} k+2 & -(k+1) \\ -(k+1) & k \end{bmatrix} = - \begin{bmatrix} k+2 & -k-1 \\ -k-1 & k \end{bmatrix} = \begin{bmatrix} -k-2 & k+1 \\ k+1 & -k \end{bmatrix}$

• NOW VERIFYING BY MULTIPLICATION

$$\begin{aligned} \underline{M} \underline{M}^{-1} &= \begin{bmatrix} k & k+1 \\ k+1 & k+2 \end{bmatrix} \begin{bmatrix} -k-2 & k+1 \\ k+1 & -k \end{bmatrix} \\ &= \begin{bmatrix} k(-k-2) + (k+1)^2 & k(k+1) - k(k+1) \\ (k+1)(-k-2) + (k+1)(k+2) & (k+1)^2 - k(k+2) \end{bmatrix} \\ &= \begin{bmatrix} \cancel{-k^2 - 2k} + \cancel{k^2 + 2k} + 1 & \cancel{k^2 + k} - \cancel{k^2 + k} \\ \cancel{-k^2 - 3k - 2} + \cancel{k^2 + 3k + 2} & \cancel{k^2 + 2k + 1} - \cancel{k^2 - 2k} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \underline{\underline{I}} \end{aligned}$$

• INDEED THE INVERSE

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1YGB, FPI PAPER M, QUESTION 2

METHOD A

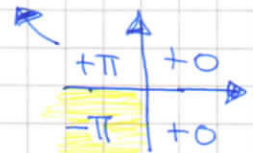
$$w = \frac{-9+3i}{1-2i} = \frac{(-9+3i)(1+2i)}{(1-2i)(1+2i)} = \frac{-9-18i+3i-6}{1+2i-2i+4}$$

$$= \frac{-15-15i}{5} = -3-3i$$

• $|w| = |-3-3i| = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$

• $\arg w = \arg(-3-3i) = \arctan\left(\frac{-3}{-3}\right) - \pi$

$$= \frac{\pi}{4} - \pi = -\frac{3\pi}{4}$$



METHOD B

• $|w| = \left| \frac{-9+3i}{1-2i} \right| = \frac{|-9+3i|}{|1-2i|} = \frac{\sqrt{81+9}}{\sqrt{1+4}} = \frac{\sqrt{90}}{\sqrt{5}}$

$$= \frac{\sqrt{5} \cdot \sqrt{2} \cdot \sqrt{9}}{\sqrt{5}} = 3\sqrt{2}$$

• $\arg w = \arg\left[\frac{-9+3i}{1-2i}\right] = \arg(-9+3i) - \arg(1-2i)$

$$= \left[\arctan\left(\frac{3}{-9}\right) + \pi \right] - \left[\arctan\left(\frac{-2}{1}\right) \right] \quad (\text{SEE ABOVE DIAGRAM})$$

$$= \pi - \arctan\frac{1}{3} + \arctan 2$$

$$= \frac{5}{4}\pi$$

→ -2π TO GET IN RANGE

$$= -\frac{3\pi}{4}$$

1YGB - FP1 PAPER M - QUESTION 3

METHOD A - USING STANDARD ROOTS RELATIONSHIPS

$$\underline{2x^2 - 8x + 9 = 0}$$

$$\bullet \alpha + \beta = -\frac{b}{a} = -\frac{-8}{2} = 4$$

$$\bullet \alpha\beta = \frac{c}{a} = \frac{9}{2}$$

PROCEED AS FOLLOWS

$$\underline{A = \alpha^2 - 1} \quad \text{and} \quad \underline{B = \beta^2 - 1}$$

$$\begin{aligned} \bullet A + B &= (\alpha^2 - 1) + (\beta^2 - 1) = \alpha^2 + \beta^2 - 2 \\ &= (\alpha + \beta)^2 - 2\alpha\beta - 2 = 4^2 - 2 \times \frac{9}{2} - 2 = 5 \end{aligned}$$

$$\begin{aligned} \bullet AB &= (\alpha^2 - 1)(\beta^2 - 1) = \alpha^2\beta^2 - \alpha^2 - \beta^2 + 1 = (\alpha\beta)^2 - (\alpha^2 + \beta^2) + 1 \\ &= \left(\frac{9}{2}\right)^2 - [(\alpha + \beta)^2 - 2\alpha\beta] + 1 = \left(\frac{9}{2}\right)^2 - [4^2 - 2 \times \frac{9}{2}] + 1 \\ &= \frac{81}{4} - 7 + 1 = \frac{57}{4} \end{aligned}$$

HENCE THE REQUIRED QUADRATIC WILL BE

$$\Rightarrow x^2 - (A+B)x + (AB) = 0$$

$$\Rightarrow x^2 - 5x + \frac{57}{4} = 0$$

$$\Rightarrow \underline{4x^2 - 20x + 57 = 0} //$$

METHOD B - BY "FOZANG" A SOLUTION

$$\underline{\text{LET } y = x^2 - 1} \Rightarrow x^2 = y + 1$$

$$\Rightarrow x = \pm\sqrt{y+1}$$

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1YGB - FPI PAPER M - QUESTION 3

SUBSTITUTE INTO THE QUADRATIC IN x

$$\Rightarrow 2(\pm\sqrt{y+1})^2 - 8(\pm\sqrt{y+1}) + 9 = 0$$

$$\Rightarrow 2(y+1) \pm 8\sqrt{y+1} + 9 = 0$$

$$\Rightarrow \pm 8\sqrt{y+1} = -9 - 2(y+1)$$

$$\Rightarrow \pm 8\sqrt{y+1} = -9 - 2y - 2$$

$$\Rightarrow \pm 8\sqrt{y+1} = -2y - 11$$

$$\Rightarrow 64(y+1) = (-2y-11)^2$$

$$\Rightarrow 64y + 64 = 4y^2 + 44y + 121$$

$$\Rightarrow 0 = 4y^2 - 20y + 57$$

or

$$\underline{4x^2 - 20x + 57 = 0}$$

↖ BFOZF

IYGB - FPI PAPER M - QUESTION 4

START BY FINDING THE INVERSE OF R - USE ELEMENTARY

ROW OPERATIONS

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{r_1 \times (-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

(SELF INVERSE)

PARAMETERIZE THE LINE

$$\frac{x+2}{3} = \frac{y-1}{2} = \frac{z-1}{4} = \lambda \Rightarrow \begin{aligned} x &= 3\lambda - 2 \\ y &= 2\lambda + 1 \\ z &= 4\lambda + 1 \end{aligned}$$

$$\Rightarrow \underline{x} = \underline{R} \underline{\alpha}$$

$$\Rightarrow \underline{R}^{-1} \underline{x} = \underline{R}^{-1} \underline{R} \underline{\alpha}$$

$$\Rightarrow \underline{\alpha} = \underline{R}^{-1} \underline{x}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3\lambda - 2 \\ 2\lambda + 1 \\ 4\lambda + 1 \end{bmatrix} = \begin{bmatrix} -3\lambda + 2 \\ 2\lambda + 1 \\ 4\lambda + 1 \end{bmatrix}$$

ELIMINATE λ TO GET

$$\frac{x-2}{-3} = \frac{y-1}{2} = \frac{z-1}{4} = \lambda$$

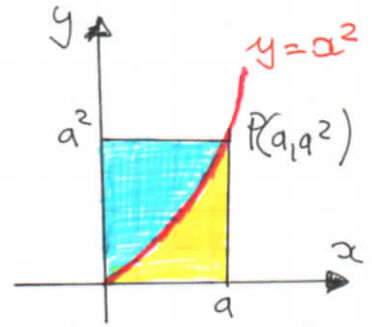
OR

$$\frac{2-x}{3} = \frac{y-1}{2} = \frac{z-1}{4}$$

YGB - FPI PAPER M - QUESTION 5

● LET P HAVE CO-ORDINATES (a, a^2) , $a > 0$

● VOLUME OF REVOLUTION ABOUT THE x AXIS



$$V_x = \pi \int_{x_1}^{x_2} y^2 dx$$

$$V_x = \pi \int_0^a (x^2)^2 dx = \pi \int_0^a x^4 dx = \frac{1}{5} \pi [x^5]_0^a = \frac{1}{5} \pi a^5$$

● VOLUME OF REVOLUTION ABOUT THE y AXIS

$$V_y = \pi \int_{y_1}^{y_2} x^2 dy$$

$$V_y = \pi \int_0^{a^2} y dy = \frac{1}{2} \pi [y^2]_0^{a^2} = \frac{1}{2} \pi a^4$$

● Now $V_x = V_y$

$$\frac{1}{5} \pi a^5 = \frac{1}{2} \pi a^4$$

$$a^5 = \frac{5}{2} a^4$$

$$a = \frac{5}{2}$$

$(a \neq 0)$

IYGB - FPI PAPER 11 - QUESTION 6

WRITE THE COMPLEX NUMBERS IN CARTESIAN FORM

$$z = x + iy \quad w = u + iv \quad \bar{w} = u - iv$$

Hence we have

$$\begin{aligned} |z+w|^2 - |z-\bar{w}|^2 &= |x+iy+u+iv|^2 - |x+iy-(u-iv)|^2 \\ &= |(x+u)+i(y+v)|^2 - |(x-u)+i(y+v)|^2 \\ &= \left[\sqrt{(x+u)^2 + (y+v)^2} \right]^2 - \left[\sqrt{(x-u)^2 + (y+v)^2} \right]^2 \\ &= (x+u)^2 + \cancel{(y+v)^2} - (x-u)^2 - \cancel{(y+v)^2} \\ &= (x+u)^2 - (x-u)^2 \\ &= (x+u+x-u)(x+u-x+u) \\ &= (2x)(2u) \\ &= 4xu \\ &= 4 \operatorname{Re} z \operatorname{Re} w \end{aligned}$$

// AS REQUIRED

ALTERNATIVE METHOD USING $z\bar{z} = |z|^2$

$$\begin{aligned} |z+w|^2 - |z-\bar{w}|^2 &= [z+w][\overline{z+w}] - [z-\bar{w}][\overline{z-\bar{w}}] \\ &= [z+w][\bar{z}+\bar{w}] - [z-\bar{w}][z-\bar{w}] \\ &= (z+w)(\bar{z}+\bar{w}) - (z-\bar{w})(z-\bar{w}) \\ &= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} - (z\bar{z} - z\bar{w} - \bar{w}z + \bar{w}w) \\ &= \cancel{z\bar{z}} + z\bar{w} + w\bar{z} + \cancel{w\bar{w}} - \cancel{z\bar{z}} + z\bar{w} + \bar{w}z - \cancel{\bar{w}w} \end{aligned}$$

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IXGB - FPI PAPER M - QUESTION 6

$$= z\bar{w} + w\bar{z} + zw + \bar{w}\bar{z}$$

$$= zw + z\bar{w} + w\bar{z} + \bar{w}\bar{z}$$

$$= z(w + \bar{w}) + \bar{z}(w + \bar{w})$$

$$= (w + \bar{w})(z + \bar{z})$$

$$= (2\operatorname{Re}w)(2\operatorname{Re}z)$$

$$= 4\operatorname{Re}w \operatorname{Re}z$$

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1YGB - FPI PAPER M - QUESTION 7

A(7,2,6) B(9,10,4) C(-3,-2,-2)

a) START BY FINDING \vec{AB} & \vec{AC}

$\vec{AB} = (9,10,4) - (7,2,6) = (2,8,-2) \sim (1,4,-1)$

$\vec{AC} = (-3,-2,-2) - (7,2,6) = (-10,-4,-8) \sim (5,2,4)$

LET THE REQUIRED VECTOR BE (p,q,r)

$$\left\{ \begin{array}{l} (p,q,r) \cdot (1,4,-1) = 0 \\ (p,q,r) \cdot (5,2,4) = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} p + 4q - r = 0 \\ 5p + 2q + 4r = 0 \end{array} \right\}$$

LET $r=1$ IN THE ABOVE EQUATIONS

$$\left\{ \begin{array}{l} p + 4q - 1 = 0 \quad \times -5 \\ 5p + 2q + 4 = 0 \quad \times 1 \end{array} \right\} \Rightarrow$$

$$\left\{ \begin{array}{l} -5p - 20q + 5 = 0 \\ 5p + 2q + 4 = 0 \end{array} \right\} \Rightarrow \begin{array}{l} -18q + 9 = 0 \\ 18q = 9 \\ q = \frac{1}{2} \end{array}$$

$$\begin{array}{l} \Rightarrow p + 4q - 1 = 0 \\ \Rightarrow p + 2 - 1 = 0 \\ \Rightarrow p = -1 \end{array}$$

HENCE A PERPENDICULAR VECTOR TO BOTH \vec{AB} & \vec{AC} IS

$(p,q,r) = (-1, \frac{1}{2}, 1) \sim (2, -1, 2)$

1YGB - EPI PAPER M - QUESTION 7

Hence an equation of the required plane is

$$2x - y - 2z = \text{constant}$$

Using the point A(7, 2, 6)

$$(2 \times 7) - 2 - (2 \times 6) = \text{constant}$$

$$\text{constant} = 0$$

\therefore $2x - y - 2z = 0$

b) THE REQUIRED LINE HAS DIRECTION VECTOR (2, -1, 2)

$$\Rightarrow \underline{r} = (\text{FIXED POINT}) + \lambda (\text{DIRECTION VECTOR})$$

$$\Rightarrow \underline{r} = (11, 3, -4) + \lambda (2, -1, 2)$$

$$\Rightarrow (x, y, z) = (2\lambda + 11, -\lambda + 3, -2\lambda - 4)$$

Solving simultaneously with $2x - y - 2z = 0$

$$\Rightarrow 2(2\lambda + 11) - (-\lambda + 3) - 2(-2\lambda - 4) = 0$$

$$\Rightarrow 4\lambda + 22 + \lambda + 3 + 4\lambda + 8 = 0$$

$$\Rightarrow 9\lambda = -27$$

$$\Rightarrow \lambda = -3$$

\therefore Q(5, 6, 2)

1YGB - FPI PAPER 11 - QUESTION 7

c) FINALLY TO FIND THE DISTANCE

$$\left. \begin{array}{l} P(11, 3, -4) \\ Q(5, 6, 2) \end{array} \right\} \Rightarrow |\vec{PQ}| = |q - p|$$
$$= |(5, 6, 2) - (11, 3, -4)|$$
$$= |(-6, 3, 6)|$$
$$= \sqrt{36 + 9 + 36}$$
$$= \sqrt{81}$$
$$= 9$$

ALTERNATIVE

SINCE |DIRECTION VECTOR| = |2, -1, -2| = 3
AND $\lambda = -3$, THE REQUIRED DISTANCE
WILL BE $3 \times |\lambda| = 9$

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IYGB - FPI PAGE 11 - QUESTION 8

$$f(n) = 4^{n+1} + 5^{2n-1}$$

• THE BASE CASE, i.e. $n=1$

$$f(1) = 4^2 + 5^1 = 16 + 5 = 21 \quad \text{i.e. DIVISIBLE BY 21}$$

• INDUCTIVE HYPOTHESIS

SUPPOSE THAT $f(n)$ IS DIVISIBLE BY 21 FOR $n=k \in \mathbb{N}$, i.e. $f(k) = 21m$ FOR SOME $m \in \mathbb{N}$

THEN

$$f(k+1) - f(k) = (4^{k+2} + 5^{2k+1}) - (4^{k+1} + 5^{2k-1})$$

$$f(k+1) - 21m = 4 \times 4^{k+1} - 4^{k+1} + 5^2 \times 5^{2k-1} - 5^{2k-1}$$

$$f(k+1) - 21m = 4 \times 4^{k+1} - 4^{k+1} + 25 \times 5^{2k-1} - 5^{2k-1}$$

$$f(k+1) - 21m = 3 \times 4^{k+1} + 24 \times 5^{2k-1}$$

BUT

$$f(k) = 4^{k+1} + 5^{2k+1} = 21m$$

$$f(k+1) - 21m = [3 \times 4^{k+1} + 3 \times 5^{2k-1}] + 21 \times 5^{2k-1}$$

$$f(k+1) - 21m = 3 \times f(k) + 21 \times 5^{2k-1}$$

$$f(k+1) = 84m + 21 \times 5^{2k-1}$$

$$f(k+1) = \underline{21} \times [4m + 5^{2k-1}]$$

• CONCLUSION

IF $f(k)$ IS DIVISIBLE BY 21 FOR $k \in \mathbb{N}$, SO IS $f(k+1)$. SINCE $f(1)$ IS DIVISIBLE BY 21 FOR ALL $n \in \mathbb{N}$

IYGB - FPI PAPER 11 - QUESTION 9

$$\begin{aligned}
 a) \quad \sum_{r=1}^n (r^3 - r) &= \sum_{r=1}^n r^3 - \sum_{r=1}^n r = \frac{1}{4}n^2(n+1)^2 - \frac{1}{2}n(n+1) \\
 &= \frac{1}{4}n(n+1)[n(n+1) - 2] = \frac{1}{4}n(n+1)(n^2+n-2) \\
 &= \frac{1}{4}n(n+1)(n-1)(n+2)
 \end{aligned}$$

b) EVALUATE IN SECTIONS

$$\begin{aligned}
 \Rightarrow \sum_{r=5}^{10} [r^3 - r + 6k] - \sum_{r=1}^{12} (r^2 + k^2) &= 70 \\
 \Rightarrow \sum_{r=5}^{10} (r^3 - r) + 6k \sum_{r=5}^{10} 1 - \sum_{r=1}^{12} r^2 - k^2 \sum_{r=1}^{12} 1 &= 70 \\
 \Rightarrow \left[\sum_{r=1}^{10} (r^3 - r) - \sum_{r=1}^4 (r^3 - r) \right] + 6k \underbrace{(1+1+1+\dots+1)}_6 - \sum_{r=1}^{12} r^2 - k^2 \underbrace{(1+\dots+1)}_{12} &= 70 \\
 \Rightarrow \frac{1}{4} \times 9 \times 10 \times 11 \times 12 - \frac{1}{4} \times 3 \times 4 \times 5 \times 6 + 6k \times 6 - \frac{1}{6} \times 12 \times 13 \times 25 - k^2 \times 12 &= 70 \\
 &\quad \quad \quad \uparrow \quad \quad \quad \uparrow \\
 &\quad \quad \quad \frac{1}{6}n(n+1)(2n+1) \\
 \Rightarrow 2970 - 90 + 36k - 650 - 12k^2 &= 70 \\
 \Rightarrow 0 = 12k^2 - 36k - 2160 \\
 \Rightarrow k^2 - 3k - 180 &= 0 \\
 \Rightarrow (k - 15)(k + 12) &= 0 \\
 \Rightarrow k = \begin{matrix} 15 \\ -12 \end{matrix}
 \end{aligned}$$