

1YGB - FPI PAPER 0 - QUESTION 1

a) CARRY OUT THE REQUIRED "MULTIPLICATIONS"

$$\underline{A}^2 = \underline{A}\underline{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 2 \times 0 & 1 \times 2 + 2 \times 1 \\ 0 \times 1 + 1 \times 0 & 0 \times 2 + 1 \times 1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}}}$$

$$\underline{A}^3 = \underline{A}^2 \underline{A} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + 4 \times 0 & 1 \times 2 + 4 \times 1 \\ 0 \times 1 + 1 \times 0 & 0 \times 2 + 1 \times 1 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}}}$$

b) A POSSIBLE FORM OF \underline{A}^n MIGHT BE

$$\underline{A}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$$

● IF $n=1$, $\underline{A}^1 = \underline{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, IE THE RESULT STANDS

● SUPPOSE THAT THE RESULT STANDS FOR $n=k \in \mathbb{N}$

$$\Rightarrow \underline{A}^k = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \underline{A}^k \underline{A} = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \underline{A}^{k+1} = \begin{pmatrix} 1 \times 1 + 2k \times 0 & 1 \times 2 + 2k \times 1 \\ 0 \times 1 + 1 \times 0 & 0 \times 2 + 1 \times 1 \end{pmatrix}$$

$$\Rightarrow \underline{A}^{k+1} = \begin{pmatrix} 1 & 2k+2 \\ 0 & 1 \end{pmatrix}$$

1 YGB - FPI PAPER 0 - QUESTION 1

$$\Rightarrow A^{k+1} = \begin{pmatrix} 1 & 2(k+1) \\ 0 & 1 \end{pmatrix}$$

● IF THE RESULT HOLDS FOR $n=k \in \mathbb{N}$, THEN IT ALSO HOLDS FOR $n=k+1$

SINCE THE RESULT HOLDS FOR $n=1$, THEN IT MUST HOLD FOR ALL $n \in \mathbb{N}$

1YGB - FPI PAGE 0 - QUESTION 2

$$z^4 - 8z^3 + 33z^2 - 68z + 52 = 0, z \in \mathbb{C}$$

AS THE EQUATION HAS REAL COEFFICIENTS, ANY ROOTS IF COMPLEX MUST EXIST AS CONJUGATE PAIRS

$$\therefore z_1 = 2 + 3i \implies z_2 = 2 - 3i$$

PROCEED AS FOLLOWS

$$\begin{aligned} (z - z_1)(z - z_2) &= [z - (2 + 3i)][z - (2 - 3i)] \\ &= [(z - 2) - 3i][(z - 2) + 3i] \\ &= (z - 2)^2 - (3i)^2 \\ &= z^2 - 4z + 4 + 9 \\ &= z^2 - 4z + 13 \end{aligned}$$

BY "LONG DIVISION" OR INSPECTION

$z^2 - 4z + 13$	$\begin{array}{r} z^2 - 4z + 4 \\ \hline z^4 - 8z^3 + 33z^2 - 68z + 52 \\ -z^4 + 4z^3 - 13z^2 \\ \hline -4z^3 + 20z^2 - 68z + 52 \\ +4z^3 - 16z^2 + 52z \\ \hline 4z^2 - 16z + 52 \\ -4z^2 + 16z - 52 \\ \hline 0 \end{array}$
-----------------	--

YGB - FPI PAGE 0 - QUESTION 2

HENCE WE HAVE

$$\begin{aligned} z^4 - 8z^3 + 33z^2 - 68z + 52 &= (z^2 - 4z + 13)(z^2 - 4z + 4) \\ &= (z^2 - 4z + 13)(z - 2)^2 \end{aligned}$$

HENCE THE FULL SET OF SOLUTIONS IS

$$z = \begin{cases} 2 + 3i & (\text{GIVEN}) \\ 2 - 3i \\ 2 & (\text{REPEATED}) \end{cases}$$

- 1 -

1YGB - FPI PAPER 0 - QUESTION 3

LET THE SMALLER ROOT OF THE QUADRATIC BE α

● THE SUM OF THE ROOTS: $\alpha + (\alpha + 3) = -\frac{b}{a} = -\frac{5}{2}$
I.E. $2\alpha + 3 = -\frac{5}{2}$
 $2\alpha = -\frac{11}{2}$
 $\alpha = -\frac{11}{4}$

● THE PRODUCT OF THE ROOTS: $\alpha(\alpha + 3) = \frac{c}{a} = \frac{c}{2}$
I.E. $c = 2\alpha(\alpha + 3)$
 $c = 2\left(-\frac{11}{4}\right)\left(-\frac{11}{4} + 3\right)$
 $c = -\frac{11}{2} \times \frac{1}{4}$
 $c = -\frac{11}{8}$

ALTERNATIVE - WITHOUT USING DIRECTLY RESULTS ON THE SUM AND PRODUCT OF ROOTS OF A QUADRATIC

● LET THE SMALLER OF THE TWO ROOTS BE α

THW $2x^2 + 5x + c = 0$
 $\Rightarrow x^2 + \frac{5x}{2} + \frac{c}{2} = 0$
 $\Rightarrow (x - \alpha)(x - (\alpha + 3)) = 0$
 $\Rightarrow x^2 - (\alpha + 3)x - \alpha x + \alpha(\alpha + 3) = 0$
 $\Rightarrow x^2 - \underline{(2\alpha + 3)x} + \underline{\alpha(\alpha + 3)} = 0$

1YGB - FPI PAPER 0 - QUESTION 3

● BY COMPARISON WE HAVE

$$\bullet \frac{5}{2} = -(2\alpha + 3)$$

$$\Rightarrow 2\alpha + 3 = -\frac{5}{2}$$

$$\Rightarrow 4\alpha + 6 = -5$$

$$\Rightarrow 4\alpha = -11$$

$$\Rightarrow \alpha = -\frac{11}{4}$$

q

$$\bullet \frac{c}{2} = \alpha(\alpha + 3)$$

$$\Rightarrow c = 2\alpha(\alpha + 3)$$

$$\Rightarrow c = 2\left(-\frac{11}{4}\right)\left(-\frac{11}{4} + 3\right)$$

$$\Rightarrow c = -\frac{11}{2} \times \frac{1}{4}$$

$$\Rightarrow c = -\frac{11}{8}$$

1YGB - FPI PAPER 0 - QUESTION 4

AS THE REVOLUTION IS ABOUT THE y AXIS, REARRANGE THE EQUATION FOR x

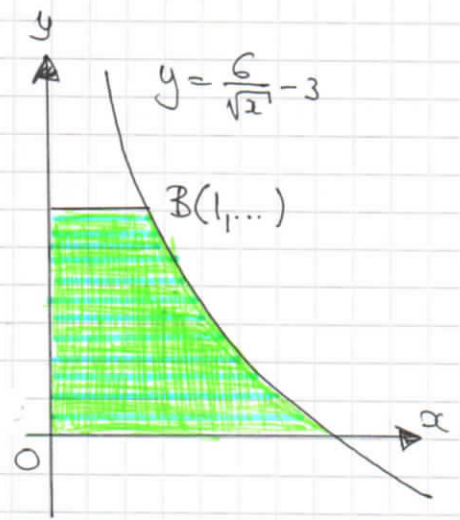
$$\Rightarrow y = \frac{6}{\sqrt{x}} - 3$$

$$\Rightarrow y + 3 = \frac{6}{\sqrt{x}}$$

$$\Rightarrow (y+3)^2 = \frac{36}{x}$$

$$\Rightarrow x = \frac{36}{(y+3)^2}$$

$$\Rightarrow x^2 = \frac{1296}{(y+3)^4}$$



BY INSPECTION THE y CO-ORDINATE OF B IS 3

THUS WE HAVE

$$V = \pi \int_{y_1}^{y_2} [x(y)]^2 dy = \pi \int_0^3 \frac{1296}{(y+3)^4} dy$$

$$= \pi \int_0^3 1296(y+3)^{-4} dy = \pi \left[\frac{1296}{-3} (y+3)^{-3} \right]_0^3$$

$$= \pi \left[-\frac{432}{(y+3)^3} \right]_0^3 = 432\pi \left[\frac{1}{(y+3)^3} \right]_3^0$$

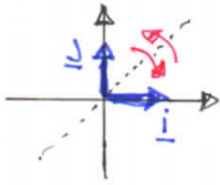
$$= 432\pi \left[\frac{1}{27} - \frac{1}{216} \right] = 432\pi \times \frac{7}{216} = 14\pi$$

AS REQUIRED

YGB - FPI PAPER 0 - QUESTION 5

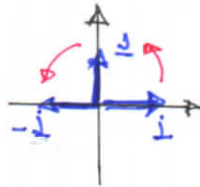
● START OBTAINING THE THREE MATRICES

REFLECTION
ABOUT $y=x$



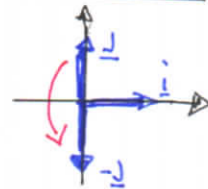
$$\underline{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

NEXT ROTATION BY 90°
ANTICLOCKWISE



$$\underline{B} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

FINALLY REFLECTION
ABOUT z AXIS



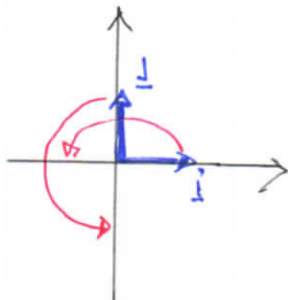
$$\underline{C} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

● MULTIPLY IN THE CORRECT ORDER

$$\begin{aligned} \underline{C}\underline{B}\underline{A} &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

$$\text{e.g. } \begin{array}{l} i \longrightarrow -i \\ j \longrightarrow -j \end{array}$$

WITH POSITIVE DETERMINANT, SO NO REFLECTION



∴ ROTATION ABOUT 0, BY 180°

IYGB - FPI PAGE 0 - QUESTION 6

TIDY UP AS FOLLOWS

$$\Rightarrow \frac{1}{x+iy} + \frac{1}{1+2i} = 1$$

$$\Rightarrow \frac{1}{x+iy} = 1 - \frac{1}{1+2i}$$

$$\Rightarrow \frac{1}{x+iy} = 1 - \frac{1-2i}{(1+2i)(1-2i)}$$

$$\Rightarrow \frac{1}{x+iy} = 1 - \frac{1-2i}{5}$$

$$\Rightarrow \frac{5}{x+iy} = 5 - (1-2i)$$

$$\Rightarrow \frac{5}{x+iy} = 4 + 2i$$

$$\Rightarrow \frac{x+iy}{5} = \frac{1}{4+2i}$$

$$\Rightarrow \frac{1}{5}(x+iy) = \frac{4-2i}{(4+2i)(4-2i)}$$

$$\Rightarrow \frac{1}{5}(x+iy) = \frac{4-2i}{16+4}$$

$$\Rightarrow \frac{1}{5}(x+iy) = \frac{4-2i}{20}$$

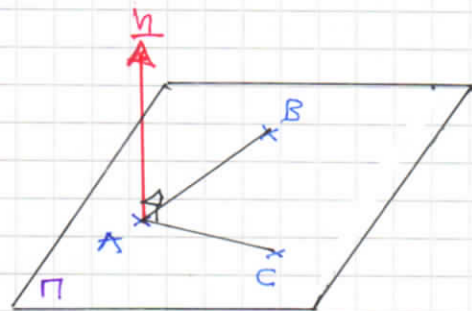
$$\Rightarrow \frac{1}{5}(x+iy) = \frac{1}{5} - \frac{1}{10}i$$

$$\Rightarrow \underline{x+iy = 1 - \frac{1}{2}i}$$

$\therefore x = 1$
 $y = -\frac{1}{2}$

1YGB - FPI PAGE 0 - QUESTION 7

a) SHOWING THAT \vec{AB} & \vec{AC} ARE BOTH PERPENDICULAR TO THE GIVEN VECTOR



• $\vec{AB} = \mathbf{b} - \mathbf{a} = (-1, -2, 0) - (1, -3, 1)$

$\vec{AB} = (-2, 1, -1)$

• $\vec{AC} = \mathbf{c} - \mathbf{a} = (0, -1, -4) - (1, -3, 1)$

$\vec{AC} = (-1, 2, -5)$

$\left(\begin{array}{l} \vec{AB} \cdot (1, 3, 1) = (-2, 1, -1) \cdot (1, 3, 1) = -2 + 3 - 1 = 0 \\ \vec{AC} \cdot (1, 3, 1) = (-1, 2, -5) \cdot (1, 3, 1) = -1 + 6 - 5 = 0 \end{array} \right)$

INDEED A NORMAL TO Π

b) THE CARTESIAN EQUATION OF Π MUST BE

$|x + 3y + z = \text{CONSTANT}$

$x + 3y + z = \text{CONSTANT}$

USING $A(1, -3, 1)$ GIVES

$1 + 3(-3) + 1 = \text{CONSTANT}$

$-7 = \text{CONSTANT}$

$\therefore x + 3y + z = -7$

$x + 3y + z + 7 = 0$

LYGB - FPI PAPER 0 - QUESTION 7c) SOLVING SIMULTANEOUSLY

$$\Gamma: x+3y+z+7=0$$

$$\Gamma = (2, 0, 1) + \lambda(5, 1, 2)$$

$$\Gamma = (5\lambda+2, \lambda, 2\lambda+1)$$

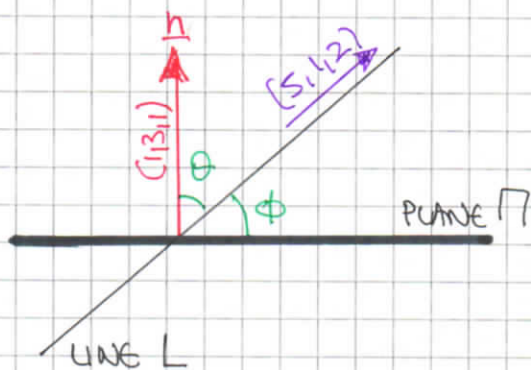
$$(5\lambda+2) + 3\lambda + (2\lambda+1) + 7 = 0$$

$$10\lambda + 10 = 0$$

$$\lambda = -1$$

$$\therefore [5(-1)+2, -1, 2(-1)+1] \text{ YIELDS } \underline{\underline{(-3, -1, -1)}}$$

d)

LOOKING AT THE DIAGRAM

$$\Rightarrow (1, 3, 1) \cdot (5, 1, 2) = |1, 3, 1| |5, 1, 2| \cos \theta$$

$$\Rightarrow 5+3+2 = \sqrt{1+9+1} \sqrt{25+1+4} \cos \theta$$

$$\Rightarrow 10 = \sqrt{11} \sqrt{30} \cos \theta$$

$$\Rightarrow \cos \theta = \frac{10}{\sqrt{11 \times 30}}$$

$$\Rightarrow \theta \approx 56.599^\circ$$

\(\therefore\) REQUIRED ANGLE IS \(\phi\)

$$\Rightarrow \phi = 90 - \theta$$

$$\Rightarrow \underline{\underline{\phi \approx 33.4^\circ}}$$

IYGB - FPI PAPER 0 - QUESTION 8

a) TRIVIALLY WE HAVE

$$u_1 = \sum_1^1 = 1^2(1+1)(1+2) = 1 \times 2 \times 3 = 6 //$$

b) i) USING $\sum_n^1 - \sum_{n-1}^1 = u_n$

$$\begin{aligned} \Rightarrow u_n &= n^2(n+1)(n+2) - (n-1)^2[(n-1)+1][(n-1)+2] \\ \Rightarrow u_n &= n^2(n+1)(n+2) - (n-1)^2 n(n+1) \\ \Rightarrow u_n &= n(n+1)[n(n+2) - (n-1)^2] \\ \Rightarrow u_n &= n(n+1)[\cancel{n^2} + 2n - (\cancel{n^2} - 2n + 1)] \\ \Rightarrow u_n &= n(n+1)(4n-1) // \text{As required} \end{aligned}$$

$$\begin{aligned} \text{ii) } \sum_{r=n+1}^{2n} u_r &= \sum_{2n}^1 - \sum_n^1 \\ &= (2n)^2(2n+1)(2n+2) - n^2(n+1)(n+2) \\ &= 4n^2(2n+1) \times 2(n+1) - n^2(n+1)(n+2) \\ &= n^2(n+1)[8(2n+1) - (n+2)] \\ &= n^2(n+1)(8n+8-n-2) \\ &= n^2(n+1)(15n+6) \\ &= \underline{3n^2(n+1)(5n+2)} // \text{As required} \end{aligned}$$

1YGB - FPI PAPER 0 - QUESTION 9

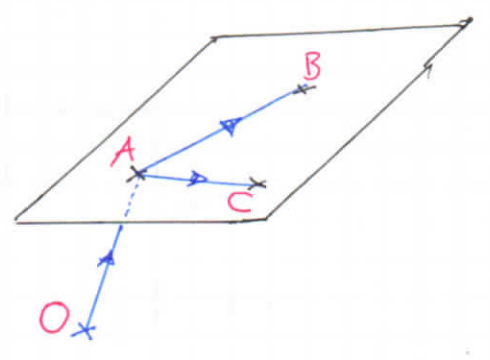
START BY PARAMETRIZING THE PLANE - TAKE ANY 3 POINTS ON THE PLANE SAY A(6,0,0), B(0,12,0) & C(0,0,-12)

$\vec{AB} = \mathbf{b} - \mathbf{a} = (0,12,0) - (6,0,0) = (-6,12,0)$ SCALE IT TO $(-1,2,0)$
 $\vec{AC} = \mathbf{c} - \mathbf{a} = (0,0,-12) - (6,0,0) = (-6,0,-12)$ SCALE IT TO $(1,0,2)$

HENCE WE HAVE

$$\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 - \lambda + \mu \\ 2\lambda \\ 2\mu \end{bmatrix}$$



Now TRANSFORM THE PARAMETRIZED PLANE

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 6 - \lambda + \mu \\ 2\lambda \\ 2\mu \end{bmatrix} = \begin{bmatrix} 6 - \lambda + \mu + 4\lambda \\ 6 - \lambda + \mu + 2\mu \\ 6 - \lambda + \mu + 2\lambda + 2\mu \end{bmatrix} = \begin{bmatrix} 6 + 3\lambda + \mu \\ 6 - \lambda + 3\mu \\ 6 + \lambda + 3\mu \end{bmatrix}$$

$X = 6 + 3\lambda + \mu \Rightarrow \underline{\mu = X - 6 - 3\lambda}$
 $Y = 6 - \lambda + 3\mu$
 $Z = 6 + \lambda + 3\mu$

SUBSTITUTING INTO THE OTHER TWO EQUATIONS

THUS $Y = 6 - \lambda + 3(X - 6 - 3\lambda)$
 $Z = 6 + \lambda + 3(X - 6 - 3\lambda)$

LYGB - FPI PAPER 0 - QUESTION 9

$$\left. \begin{aligned} Y &= 6 - \lambda + 3X - 18 - 9\lambda \\ Z &= 6 + \lambda + 3X - 18 - 9\lambda \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} Y &= 3X - 12 - 10\lambda \\ Z &= 3X - 12 - 8\lambda \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} 10\lambda &= 3X - Y - 12 \\ 8\lambda &= 3X - Z - 12 \end{aligned} \right\} \Rightarrow$$

$$\left. \begin{aligned} 40\lambda &= 12X - 4Y - 48 \\ 40\lambda &= 15X - 5Z - 60 \end{aligned} \right\} \Rightarrow$$

$$\Rightarrow 12X - 4Y - 48 = 15X - 5Z - 60$$

$$\Rightarrow -3X - 4Y + 5Z = -12$$

$$\Rightarrow \underline{3X + 4Y + 5Z = 12}$$