IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper D

Difficulty Rating: 3.7525

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

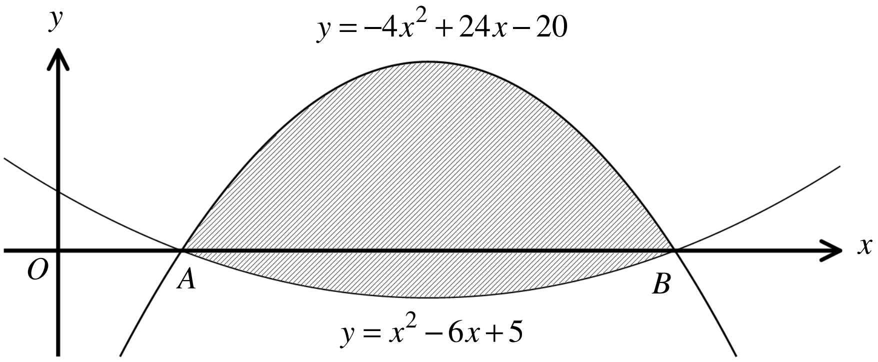
This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 22 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1



The figure above shows the graph of the curves with equations

$$y=-4x^2+24x-20 \quad \text{and} \quad y=x^2-6x+5.$$

The two curves intersect each other at the points A and B.

The finite region R bounded by the two curves is shown shaded in the figure.

Find the exact area of R. **(8)**

Question 2

$$f(x)=|x-80|, \; x\in\mathbb{R}.$$

a) Solve the inequality

$$f(x)<10.$$ **(3)**

b) Find the value of the integer n, such that $f\left(1.2^n\right)<10$. **(4)**

Question 3

$$2x^2 - xy - y^2 .$$

Factorize the above quadratic expression.

You may factorize by inspection, or by using the quadratic formula or by completing the square. **(3)**

Question 4

$$f(x) = \frac{4+x}{(1+3x)^2}, \quad |x| < \frac{1}{3} .$$

a) Find the series expansion of $(1+3x)^{-1}$, up and including the term in x^3. **(3)**

b) By differentiating both sides of the expansion found in part **(a)**, show that

$$(1+3x)^{-2} = 1 - 6x + 27x^2 + \dots$$ **(3)**

c) Hence find the first three terms in the series expansion of $f(x)$. **(4)**

Question 5 (***)

The sixteenth term of an arithmetic series is 6.

The sum of the first sixteen terms is 456.

a) Find the first term and the common difference of the series. **(5)**

The sum of the first k terms of the series is zero.

b) Determine the value of k. **(4)**

Question 6

The number, x thousands, of reported cases of an infectious disease, t months after it was first reported, is now dropping. The rate at which it is dropping is proportional to the square of the number of the reported cases.

It is assumed that x can be treated as a continuous variable.

a) Form a differential equation in terms of x, t and a proportionality constant k. **(2)**

Initially there were 2500 reported cases and one month later they had dropped to 1600 cases.

b) Solve the differential equation to show that

$$x = \frac{40}{9t+16}.$$ **(7)**

c) Find after how many months there will be 250 reported cases. **(2)**

Question 7

$$\left(125^{\frac{1}{3}} \times 25^{\frac{1}{2}} + 16^{\frac{3}{4}} \times 64^{\frac{1}{3}} + \frac{1}{49^{-\frac{1}{2}}}\right)^{-\frac{2}{3}}.$$

Evaluate the above indicial expression, giving the final answer as a simplified fraction.

You may not use any calculating aid in the above question, and detailed workings must support the answer. **(6)**

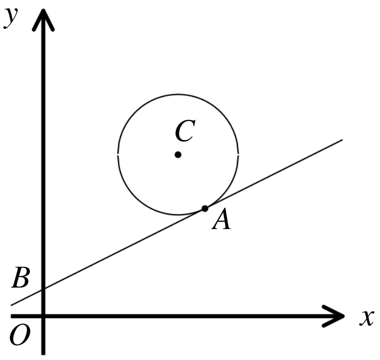
Question 8

An **even** function f, of period 2 is defined by

$$f(x) \equiv \begin{cases} 4x^2 & 0 \le x \le \frac{1}{2} \\ 1 & \frac{1}{2} \le x \le 1 \end{cases}$$

Sketch the graph of $f(x)$ for $-3 \le x \le 3$. **(4)**

Question 9



The figure above shows a circle with centre at C with equation

$$x^2 + y^2 - 10x - 12y + 56 = 0.$$

The tangent to the circle at the point $A(6,4)$ meets the y axis at the point B.

a) Find an equation of the tangent to the circle at A. **(6)**

b) Determine the area of the triangle ABC. **(3)**

Question 10

A geometric series, whose terms alternate in sign, has its first term denoted by a and its common ratio denoted by r.

The sum of the first n terms of the series is denoted by S_n.

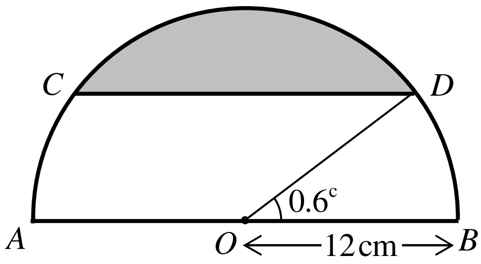
It is given that

$$S_4 = 5S_2.$$

a) Find the value of r. **(7)**

b) Given further that the fifth term of the series is 36, determine the value of a. **(2)**

Question 11



The figure above shows a semi circle with centre at O and radius 12 cm.

The diameter of the semicircle is AOB, the chord CD is parallel to AOB.

It is further given that the angle DOB is 0.6^{c}.

a) Find the area of the shaded segment. **(6)**

b) Determine the perimeter of the shaded segment. **(5)**

Question 12

The curve C has equation

$$y = 4x^2 - 7x + 11.$$

The straight line L has equation

$$y = 5x + k,$$

where k is a constant.

Given that C and L intersect at two distinct points, show that $k > 2$. **(6)**

Question 13

The following information is given for a polynomial $f(x)$.

- When $f(x)$ is divided by $(x-2)$ the remainder is 5.
- When $f(x)$ is divided by $(x+2)$ the remainder is -11.
- When $f(x)$ is divided by $(x+2)(x-2)$ the remainder is $ax+b$, and the quotient is $g(x)$, where a and b are constants.

a) Determine the value of a and the value of b. **(4)**

It is further given that

$$f(x) = 3x^4 + px + q,$$

where p and q are constants.

b) Find a simplified expression for $g(x)$. **(6)**

Question 14

By using the substitution $u^2 = e^x - 1$, or otherwise, find

$$\int_{\ln 2}^{\ln 5} \frac{3e^{2x}}{\sqrt{e^x - 1}}\, dx.$$ **(6)**

Question 15

The curve C has equation

$$y = \frac{2x^2 - 1 - 2\ln x^x}{x}, \quad x > 0.$$

The curve has a point of inflection at P.

Show that the straight line with equation $y = x$ is a tangent to C at P. **(8)**

Question 16

The points A and B have coordinates $(1, 4\sqrt{3})$ and $(-3+\sqrt{3}, 3)$, respectively.

a) Find an equation for the straight line L which passes through A and B, giving the answer in the form $y = \sqrt{3}(x+k)$, where k is an integer. **(4)**

L meets the x axis at the point C.

b) Determine the length of AC. **(3)**

c) Calculate the acute angle between L and the x axis. **(3)**

Question 17

The point $P(20,60)$ lies on a curve with parametric equations

$$x = 2at, \quad y = 8at - at^2, \quad t \in \mathbb{R}, \quad t \geq 0,$$

where a is a non zero constant.

a) Find the value of a. **(5)**

b) Determine a Cartesian equation of the curve. **(3)**

The above set of parametric equations represents the path of a golf ball, t seconds after it was struck from a fixed point on the ground, O.

The horizontal distance from O is x metres and the vertical distance above the ground level is y metres.

The ball hits the lowest point of a TV airship, which was recording the golf tournament from the air.

c) Assuming that the ground is level and horizontal, find the greatest possible height of the airship from the ground. **(4)**

Question 18

$$f(\theta) \equiv 5\cos\theta - 12\sin\theta, \ \theta \in \mathbb{R}.$$

a) Express $f(\theta)$ in the form $R\cos(\theta+\alpha)$, $R>0$, $0<\alpha<\frac{1}{2}\pi$.
Give the value of α correct to 3 decimal places. **(3)**

b) State the maximum value of $f(\theta)$ and find the smallest positive value of θ for which this maximum occurs. **(3)**

The pressure P, in suitable units, in a nuclear plant is modelled by the equation

$$P = 20 + 5\cos\left(\frac{4\pi t}{25}\right) - 12\sin\left(\frac{4\pi t}{25}\right), \quad 0 \le t < 12,$$

where t is the time in hours measured from midnight.

c) State the maximum pressure in the plant and the value of t when this maximum pressure occurs. **(3)**

d) Find the times, to the nearest minute, when $P = 15$. **(6)**

Question 19

Solve each of the following trigonometric equations.

i. $6\tan x = \dfrac{2-3\sec^2 x}{\tan x - 1}$, $\quad 0 \le x < 2\pi$, $x \ne \dfrac{\pi}{4}, \dfrac{5\pi}{4}$. **(7)**

ii. $\cos(3\theta - 60°) = \cos(3\theta + 30°)$, $\quad 0 \le \theta < 180°$. **(7)**

Question 20

A curve C has equation

$$y = 3^x + 1, \ x \in \mathbb{R}.$$

a) Sketch the graph of C, clearly indicating the equation of the asymptote to the curve and the coordinates of any intercepts with the coordinate axes. **(3)**

b) Find an equation of the curve which is obtained by reflecting the graph of C in the x axis followed by reflection of the graph of C in the y axis. **(2)**

c) Describe fully a sequence of two transformations which map the graph of C onto the graph with equation

$$y = 3^{x+1} + 3, \ x \in \mathbb{R}. \quad \textbf{(2)}$$

d) Describe fully a **single** transformation which map the graph of C onto the graph with equation

$$y = 3^{x+1} + 3, \ x \in \mathbb{R}. \quad \textbf{(2)}$$

Question 21

$$\frac{4t^2}{t-1} \equiv At + B + \frac{C}{t-1}.$$

a) Determine the value of each of the constants A, B and C. **(3)**

b) Use the substitution $t = x^{\frac{1}{4}}$ to show

$$\int_{16}^{81} \frac{1}{x^{\frac{1}{2}} - x^{\frac{1}{4}}} \, dx = 14 + 4\ln 2. \quad \textbf{(6)}$$

Question 22

The population P of seals on an island obeys the equation

$$P = \frac{800k\,e^{0.25t}}{1 + k\,e^{0.25t}}, \quad t \geq 0,$$

where k is a positive constant and t is the time, in years, measured from a certain instant.

Initially there were 175 seals on the island.

a) Find, showing a detailed method, …

i. … the value of t when the number of seals reaches 560. **(8)**

ii. …. the long term prospects of the population of these seals. **(1)**

b) Show further that

$$\frac{dP}{dt} = \frac{P(800 - P)}{3200}.$$ **(5)**

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