IYGB GCE

Mathematics SYN

Advanced Level

Synoptic Paper U

Difficulty Rating: 4.0650/0.7235

Time: 3 hours

Candidates may use any calculator allowed by the regulations of this examination.

Information for Candidates

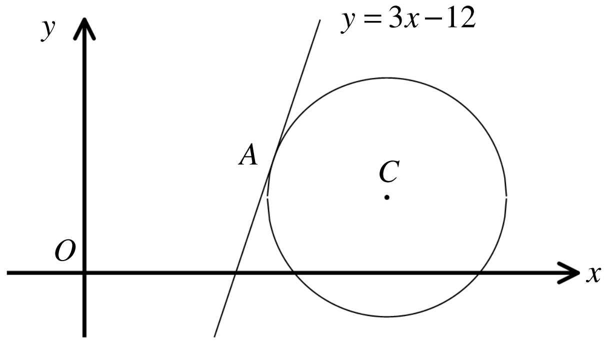
This synoptic practice paper follows closely the Advanced Level Pure Mathematics Syllabus, suitable for first assessment Summer 2018.

The standard booklet "Mathematical Formulae and Statistical Tables" may be used.
Full marks may be obtained for answers to ALL questions.
The marks for the parts of questions are shown in round brackets, e.g. **(2)**.
There are 22 questions in this question paper.
The total mark for this paper is 200.

Advice to Candidates

You must ensure that your answers to parts of questions are clearly labelled.
You must show sufficient working to make your methods clear to the Examiner.
Answers without working may not gain full credit.
Non exact answers should be given to an appropriate degree of accuracy.
The examiner may refuse to mark any parts of questions if deemed not to be legible.

Question 1



The figure above shows a circle whose centre is at $C(8, k)$, where k is a constant.

The straight line with equation

$$y = 3x - 12$$

is a tangent to the circle at the point $A(5, 3)$.

a) Find an equation of the normal to the circle at A. **(4)**

b) Determine an equation for the circle. **(5)**

Question 2

$$I = \int_0^{\sqrt{2}} \left(x^3 + x\right) e^{x^2} \, dx .$$

Show clearly that

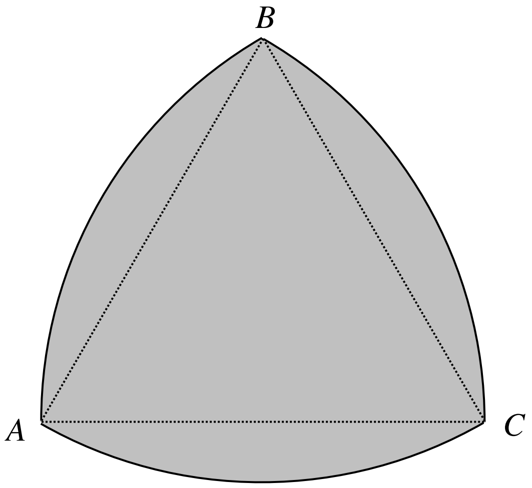
$$I = e^2 .$$ **(9)**

Question 3

Find the solutions for the following equation.

$$\left(2x^2-7x+4\right)^{x^2+2x-8}=1. \qquad (5)$$

Question 4



The figure above shows a "curved triangle", known as a Reuleaux triangle, which is constructed as follows.

Starting with an equilateral triangle ABC of side length 2 cm, a circular arc BC is drawn with centre at A. Two more circular arcs AB and AC are drawn with respective centres at C and B.

Show that the area of this Reuleaux triangle is

$$2\left(\pi-\sqrt{3}\right)\text{ cm}^2. \qquad (6)$$

Question 5

Consider the following sequence of transformations T_1, T_2 and T_3.

$$\frac{1}{x} \xrightarrow{T_1} -\frac{1}{x} \xrightarrow{T_2} -\frac{1}{x+1} \xrightarrow{T_3} 2-\frac{1}{x+1}$$ **(6)**

a) Describe geometrically the transformations T_1, T_2 and T_3.

b) Hence sketch the graph of

$$y = 2-\frac{1}{x+1}, \quad x \neq -1.$$

Indicate clearly any asymptotes and the coordinates of any intersections with the coordinate axes. **(4)**

c) Solve the equation

$$2-\frac{1}{x+1} = \frac{1}{x}.$$ **(4)**

Question 6

A quintic polynomial is defined, in terms of the constants a and b, by

$$f(x) = x^5 + ax^4 + bx^3 - x^2 + 4x - 3.$$

When $f(x)$ is divided by $(x-2)$ the remainder is -7.

When $f(x)$ is divided by $(x+1)$ the remainder is -16.

a) Determine in any order the value of a and the value of b. **(5)**

b) Find the remainder when $f(x)$ is divided by $(x-2)(x+1)$. **(5)**

Question 7

Find the possible roots of the following quadratic equation

$$mx^2 - 4x + m = 3,$$

where m is a non zero constant, given that it has repeated roots. **(7)**

Question 8

The curve C has equation

$$y = 2\left|x^2 - 6x + 8\right| \ , \ x \in \mathbb{R}.$$

The straight line L has equation

$$y = 3x - 9, \ x \in \mathbb{R}.$$

a) Sketch in the same diagram the graph of C and L.
Mark clearly in the sketch the coordinates of any x or y intercepts. **(6)**

b) Solve the equation

$$2\left|x^2 - 6x + 8\right| = 3x - 9. \quad \textbf{(6)}$$

c) Hence or otherwise, solve the inequality

$$2\left|x^2 - 6x + 8\right| > 3x - 9. \quad \textbf{(2)}$$

Question 9

Prove, with detailed workings, that

$$2\arcsin\left(\frac{2}{3}\right) = \arccos\left(\frac{1}{9}\right). \quad \textbf{(6)}$$

Question 10

A certain type of plastic sheet blocks 7% of the sunlight.

It is required to block at least 95% of the sunlight by placing N of these plastic sheets on top of each other.

Use algebra, to determine the least value of N. **(6)**

Question 11

A particle is moving on the curve with equation

$$y = 2\arcsin 3x, \quad -\tfrac{1}{3} \le x \le \tfrac{1}{3}.$$

The particle has coordinates (x, y) at time t.

When the y coordinate of the particle is $\tfrac{1}{3}\pi$, the rate at which the y coordinate is changing with time t is 2.

Find the rate at which the x coordinate of the particle changes with time, at that instant. **(8)**

Question 12

The functions f and g are defined by

$$f(x) = \sqrt{x+4}, \ x \in \mathbb{R}, \ x \ge -3$$

$$g(x) = 2x^2 - 3, \ x \in \mathbb{R}, \ x \le 47 \ .$$

a) Find a simplified expression for $gf(x)$. **(2)**

b) Determine the domain and range of $gf(x)$. **(4)**

c) Solve the equation

$$fg(x) = 17 \ .$$ **(5)**

Question 13

The finite region R is bounded by the curve with equation $y=\ln(x+3)$, the y axis and the straight line with equation $y=1$.

Show, with detailed workings, that the area of R is

$$\mathrm{e} - 6 + 3\ln 3 . \qquad (9)$$

Question 14

It is given that $\sin 1^{c} \approx 0.8415$ and $\cos 1^{c} \approx 0.5403$.

Show that $\sin\left(1.01^{c}\right) = 0.847$, correct to three decimal places. **(3)**

Question 15

Find as exact simplified surds the coordinates of the point of intersection between the graphs of

$$\sqrt{x} = 2y+3 \quad \text{and} \quad 2x+\sqrt{x}-2y\sqrt{x} = 8 . \qquad (8)$$

Question 16

Find, in exact form if appropriate, the solution of the following equation.

$$\mathrm{e}^{\frac{3}{2}x} = \mathrm{e}^{3x} - 2 . \qquad (6)$$

Question 17

Use the substitution $u = \sin x + x\tan x$ to find a simplified expression for

$$\int \frac{2x+\sin 2x+2\cos^{3}x}{(x+\cos x)\sin 2x}\, dx . \qquad (8)$$

Question 18

The height of tide, h meters, in a harbour on a certain day can be modelled by

$$h(t) = 10 + \sqrt{3}\sin(30t)° + \cos(30t)°, \ 0 \le t \le 24,$$

where t is the time in hours since midnight.

a) Find the time when the high tide and the low tide occur during the morning hours of that day and state the corresponding depth of water in the harbour at these times. **(7)**

The depth of water in this harbour needs to be **at least** 8.5 metres for a boat to dock.

A boat arrives outside the harbour at **high tide** and needs **five hours** to unload.

b) Show that the boat **has to wait** until $09:23$ to enter the harbour. **(6)**

Question 19

The temperature in a bathroom is maintained at the constant value of 20°C and the water in a hot bath is left to cool down.

The rate, in °C per second, at which the temperature of the water in the bath, T °C, is cooling down, is proportional to the difference in the temperature between the bathwater and the room.

Initially the bathwater had a temperature of 40°C, and at that instant was cooling down at the rate of 0.005°C per second.

Let t be the time in seconds, since the bathwater was left to cool down.

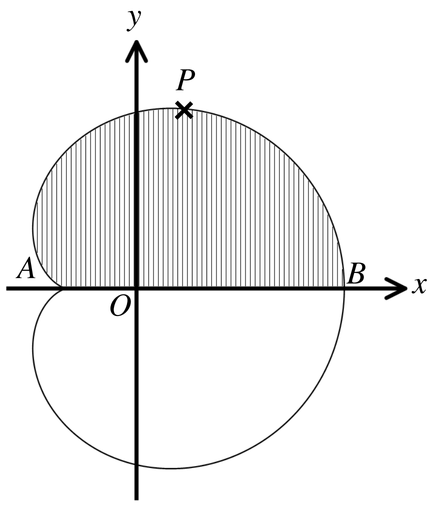
a) Show that

$$\frac{dT}{dt} = -\frac{1}{4000}(T - 20).$$

(3)

b) Solve the differential equation of part **(a)**, to find, correct to the nearest minute, after how long the temperature of the bathwater will drop to 36°C. **(7)**

Question 20



The figure above shows a curve known as a Cardioid which is symmetrical about the x axis.

The curve crosses the x axis at the points $A(-2,0)$ and $B(6,0)$.

The point P is the maximum point of the curve.

The parametric equations of this Cardioid are

$$x = 4\cos\theta + 2\cos 2\theta, \quad y = 4\sin\theta + 2\sin 2\theta, \quad 0 \le \theta < 2\pi .$$

a) Find a simplified expression for $\frac{dy}{dx}$, in terms of θ, and hence find the exact coordinates of P. **(6)**

b) Show that the area of the top half of this Cardioid, shown shaded in the figure, is given by the integral

$$\int_0^{\pi} 16\sin^2\theta + 24\sin\theta\sin 2\theta + 8\sin^2 2\theta \;\; d\theta ,$$

and hence find the exact value of the area enclosed by the Cardioid. **(8)**

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Question 21

The function f is defined as

$$f(x) \equiv \frac{ax+b}{(1-x)(1+2x)}, \; x \in \mathbb{R}, \; |x| < \frac{1}{2},$$

where a and b are constants.

a) Find the values of the constants P and Q in terms of a and b, given that

$$f(x) \equiv \frac{P}{(1-x)} + \frac{Q}{(1+2x)}.$$ **(4)**

The binomial series expansion of $f(x)$, up and including the term in x^3 is

$$f(x) = 1 + 13x + Ax^2 + Bx^3 + \ldots,$$

where A and B are constants.

b) Determine the value of the constants …

i. … a and b.

ii. … A and B. **(8)**

Question 22

A curve C has equation

$$y = \frac{2x+3}{\sqrt{2x-1}}, \; x \in \mathbb{R}, \; x > \frac{1}{2}.$$

Find the coordinates of the stationary point of C, further determining the nature of this point.

You may not use the product rule, the quotient rule or logarithmic differentiation in this question. **(12)**
