Sequences and series 3E

1 a i $r = 0.1$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{1}{1-0.1} = \frac{10}{9}$

b $r = 2$ so the series is not convergent as $|r| \geqslant 1$.

c i $r = -0.5$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{10}{1+0.5} = \frac{20}{3} = 6\frac{2}{3}$

d This is an arithmetic series and so does not converge.

e $r = 1$ so the series is not convergent as $|r| \geqslant 1$.

f i $r = \frac{1}{3}$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{3}{1-\frac{1}{3}} = \frac{9}{2} = 4\frac{1}{2}$

g This is an arithmetic series and so does not converge.

h i $r = 0.9$ so the series is convergent as $|r| < 1$.

ii $S_\infty = \frac{9}{1-0.9} = 90$

2 $a = 10,\ S_\infty = 30$

$\frac{10}{1-r} = 30$

$10 = 30(1 - r)$

$30r = 20$

$r = \frac{2}{3}$

3 $a = -5,\ S_\infty = -3$

$\frac{-5}{1-r} = -3$

$-5 = -3(1 - r)$

$3r = -2$

$r = -\frac{2}{3}$

4 $S_\infty = 60,\ r = \frac{2}{3}$

$\frac{a}{1-\frac{2}{3}} = 60$

$\frac{a}{\frac{1}{3}} = 60$

$a = 20$

5 $S_\infty = 10,\ r = -\frac{1}{3}$

$\frac{a}{1+\frac{1}{3}} = 10$

$\frac{a}{\frac{4}{3}} = 10$

$a = \frac{40}{3} = 13\frac{1}{3}$

6 $0.\dot{2}\dot{3}\ldots = \frac{23}{100} + \frac{23}{10\,000} + \frac{23}{1\,000\,000} + \ldots$

$\xrightarrow{\times\frac{1}{100}}$ $\xrightarrow{\times\frac{1}{100}}$

This is an infinite geometric series:
$a = \frac{23}{100}$ and $r = \frac{1}{100}$.

Use $S_\infty = \frac{a}{1-r}$.

$0.\dot{2}\dot{3}\ldots = \frac{\frac{23}{100}}{1-\frac{1}{100}} = \frac{\frac{23}{100}}{\frac{99}{100}}$

$= \frac{23}{100} \times \frac{100}{99} = \frac{23}{99}$

7 $S_3 = 9,\ S_\infty = 8$

$S_3 = \frac{a(1-r^3)}{1-r} = 9$ (**1**)

$S_\infty = \frac{a}{1-r} = 8$ (**2**)

$8(1-r^3) = 9$ (substituting (**2**) into (**1**))

$1-r^3 = \frac{9}{8}$

$r^3 = -\frac{1}{8}$

$r = -\frac{1}{2}$

$a = 8\left(1+\frac{1}{2}\right)$ (from (**2**))

$a = 12$

8 **a** $a = 1, r = -2x$

As the series is convergent, $|-2x| < 1$

If $x < 0$ then $2x < 1$, so $x < \frac{1}{2}$;

if $x > 0$ then $-2x < 1$, so $x > -\frac{1}{2}$

Hence, $-\frac{1}{2} < x < \frac{1}{2}$.

b $S_\infty = \frac{1}{1+2x}$

9 **a** $a = 2,\ S_\infty = 16 \times S_3$

$S_3 = \frac{2(1-r^3)}{1-r}$

$16 \times \frac{2(1-r^3)}{1-r} = \frac{2}{1-r}$

$32(1-r^3) = 2$

$r^3 = \frac{15}{16}$

$r = 0.9787$

b $u_4 = ar^3 = 2 \times 0.9787^3 = 1.875$

10 **a** $a = 30,\ S_\infty = 240$

$\frac{30}{1-r} = 240$

$\frac{1}{8} = 1-r$

$r = \frac{7}{8}$

b $u_4 - u_5 = ar^3 - ar^4$

$= 30\left(\frac{7}{8}\right)^3 - 30\left(\frac{7}{8}\right)^4$

$= 2.51$

c $S_4 = \frac{30\left(1-\left(\frac{7}{8}\right)^4\right)}{1-\frac{7}{8}}$

$= 99.3$

10 d If $S_n = \dfrac{30\left(1-\left(\frac{7}{8}\right)^n\right)}{1-\frac{7}{8}} = 180$

$$\frac{30\left(1-\left(\frac{7}{8}\right)^n\right)}{\frac{1}{8}} = 180$$

$$1-\left(\frac{7}{8}\right)^n = 0.75$$

$$0.875^n = 0.25$$

$$n = \frac{\log 0.25}{\log 0.875}$$

$n = 10.38$

$n = 11$

11 a $ar = \dfrac{15}{8}$, $S_\infty = 8$

$$\frac{a}{1-r} = 8$$

$a = 8(1 - r)$

$$a = \frac{15}{8r}$$

$$\frac{15}{8r} = 8(1-r)$$

$15 = 64r - 64r^2$

$64r^2 - 64r + 15 = 0$

b $(8r - 3)(8r - 5) = 0$

$r = \dfrac{3}{8}$ or $r = \dfrac{5}{8}$

c When $r = \dfrac{3}{8}$

$a = 8\left(1-\dfrac{3}{8}\right) = 5$

When $r = \dfrac{5}{8}$

$a = 8\left(1-\dfrac{5}{8}\right) = 3$

d $r = \dfrac{3}{8}$

If $S_n = \dfrac{5\left(1-\left(\frac{3}{8}\right)^n\right)}{1-\frac{3}{8}} = 7.99$

$$\frac{5\left(1-\left(\frac{3}{8}\right)^n\right)}{\frac{5}{8}} = 7.99$$

$$1-0.375^n = 0.99875$$

$$0.375^n = 0.00125$$

$$n = \frac{\log 0.00125}{\log 0.375}$$

$n = 6.815$

$n = 7$

Challenge

a First series: $a + ar + ar^2 + ar^3 + \ldots$
Second series: $a^2 + a^2r^2 + a^2r^4 + \ldots$
Second series has first term a^2 and common ratio r^2 so is a geometric series.

b For the first series: $S_\infty = 7$

$$\frac{a}{1-r} = 7$$

$a = 7(1 - r)$

For the second series: $S_\infty = 35$

$$\frac{a^2}{1-r^2} = 35$$

$$\frac{a^2}{(1-r)(1+r)} = 35$$

$$\frac{49(1-r)^2}{(1-r)(1+r)} = 35$$

$$49 - 49r = 35 + 35r$$

$14 = 84r$, so $r = \dfrac{1}{6}$