Conic Sections 1 Mixed Exercise 2

1 a A general parabola with equation $y^2 = 4ax$ has focus $(a, 0)$

Here $y^2 = 12x \Rightarrow 4a = 12 \Rightarrow a = \frac{12}{4} = 3$

So the focus S, has coordinates $(3, 0)$

b Line l: $y = 3x$ **(1)**

Parabola C: $y^2 = 12x$ **(2)**

Substituting **(1)** into **(2)** gives

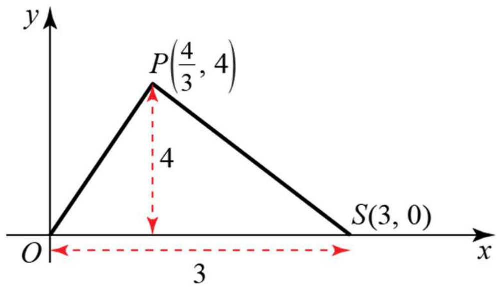
$$(3x)^2 = 12x$$
$$9x^2 = 12x$$
$$9x^2 - 12x = 0$$
$$3x(3x - 4) = 0$$
$$x = 0 \text{ or } \frac{4}{3}$$

Substituting these values of x back into equation **(1)**:

$$x = 0 \Rightarrow y = 3(0) = 0 \Rightarrow (0,\ 0)$$
$$x = \frac{4}{3} \Rightarrow y = 3\left(\frac{4}{3}\right) = 4 \Rightarrow \left(\frac{4}{3},\ 4\right)$$

As $y > 0$ at P, the coordinates of P are $\left(\frac{4}{3}, 4\right)$

1 c



$$\text{Area } \Delta OPS = \frac{1}{2}(3)(4)$$
$$= \frac{1}{2}(12)$$
$$= 6$$

Therefore, Area $\Delta OPS = 6$

2 a $(k, 6)$ lies on $y^2 = 24x$ gives

$$6^2 = 24k \Rightarrow 36 = 24k \Rightarrow \frac{36}{24} = k \Rightarrow k = \frac{3}{2}$$

b A general parabola with equation $y^2 = 4ax$ has focus $(a, 0)$

Here $y^2 = 24x \Rightarrow 4a = 24 \Rightarrow a = \frac{24}{4} = 6$

So the focus S has coordinates $(6, 0)$

c The points P and S have coordinates $P\left(\frac{3}{2}, 6\right)$ and $S(6, 0)$

$$m_l = m_{PS} = \frac{0 - 6}{6 - \frac{3}{2}} = \frac{-6}{\frac{9}{2}} = -\frac{12}{9} = -\frac{4}{3}$$

l is the line

$$y - 0 = -\frac{4}{3}(x - 6)$$
$$3y = -4(x - 6)$$
$$3y = -4x + 24$$
$$4x + 3y - 24 = 0$$

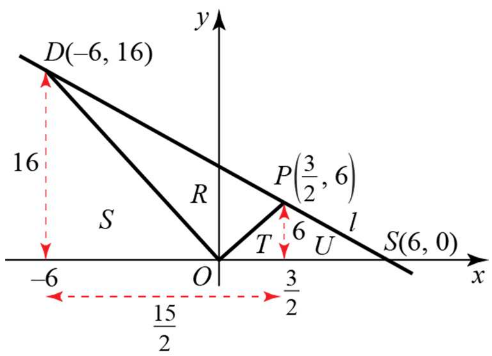
Therefore an equation for l is $4x + 3y - 24 = 0$

2 d From part **b**, as $a = 6$, an equation for the directrix is $x+6=0$ or $x=-6$

Substituting $x=-6$ into l gives:

$$4(-6)+3y-24=0$$
$$3y=24+24$$
$$3y=48$$
$$y=16$$

Hence the coordinates of D are $(-6, 16)$



Using the sketch and the regions as labelled you can find the area required.
Let Area ΔOPD = Area (R)

Method 1

$$\text{Area}(R) = \text{Area}(R+S+T) - \text{Area}(S) - \text{Area}(T)$$
$$= \frac{1}{2}(16+6)\left(\frac{15}{2}\right) - \frac{1}{2}(6)(16) - \frac{1}{2}\left(\frac{3}{2}\right)(6)$$
$$= \frac{1}{2}(22)\left(\frac{15}{2}\right) - 3(16) - \left(\frac{3}{2}\right)(3)$$
$$= \left(\frac{165}{2}\right) - 48 - \left(\frac{9}{2}\right)$$
$$= 30$$

Therefore, Area $\Delta OPD = 30$

Method 2

$$\text{Area}(R) = \text{Area}(R+S+T+U) - \text{Area}(S) - \text{Area}(TU)$$
$$= \frac{1}{2}(12)(16) - \frac{1}{2}(6)(16) - \frac{1}{2}(6)(6)$$
$$= 96 - 48 - 18$$
$$= 30$$

Therefore, Area $\Delta OPD = 30$

3 a $y = 24t$

So $t = \dfrac{y}{24}$ **(1)**

$x = 12t^2$ **(2)**

Substitute **(1)** into **(2)**:

$$x = 12\left(\frac{y}{24}\right)^2$$

So $x = \dfrac{12y^2}{576}$ simplifies to $x = \dfrac{y^2}{48}$

Hence, the Cartesian equation of C is $y^2 = 48x$

b A general parabola with equation $y^2 = 4ax$ has directrix $x + a = 0$

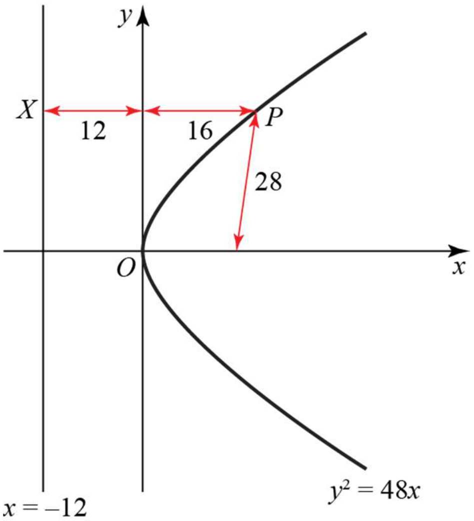
Here $y^2 = 48x \Rightarrow 4a = 48$,

giving $a = \dfrac{48}{4} = 12$

Therefore and equation of the directrix of C is $x + 12 = 0$ or $x = -12$

3 c By part **b**, since $a = 12$, the coordinates of S, the focus of C, are (12, 0)

Hence, drawing a sketch gives,



The (shortest) distance of P to the line $x = -16$ is the distance XP.

The distance $SP = 28$

The focus directrix property implies that $SP = XP = 28$

The directrix has equation $x = -12$

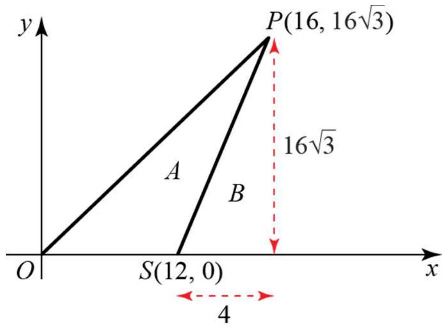
Therefore the x-coordinate of P is $x = 28 - 12 = 16$

When $x = 16, y^2 = 48(16) \Rightarrow y^2 = 3(16)^2$

As $y > 0$ then $y = \sqrt{3(16)^2} = 16\sqrt{3}$

Hence the exact coordinates of P are $(16, 16\sqrt{3})$

3 d



Let Area ΔOSP = Area(A)

$$\text{Area}(A) = \text{Area}(A+B) - \text{Area}(B)$$
$$= \frac{1}{2}(16)(16\sqrt{3}) - \frac{1}{2}(4)(16\sqrt{3})$$
$$= 128\sqrt{3} - 32\sqrt{3}$$
$$= 96\sqrt{3}$$

Area $\Delta OSP = 96\sqrt{3}$ and $k = 96$

4 a Line: $4x - 9y + 32 = 0$ **(1)**

Parabola C: $y^2 = 16x$ **(2)**

Multiplying **(1)** by 4 gives

$16x - 36y + 128 = 0$ **(3)**

Substituting **(2)** into **(3)** gives

$$y^2 - 36y + 128 = 0$$
$$(y-4)(y-32) = 0$$
$$y = 4, 32$$

When $y = 4$,

$$4^2 = 16x \Rightarrow x = \frac{16}{16} = 1$$
$$\Rightarrow (1, 4).$$

When $y = 32$,

$$32^2 = 16x \Rightarrow x = \frac{1024}{16} = 64$$
$$\Rightarrow (64, 32)$$

The coordinates of P and Q are (1, 4) and (64, 32)

4 b $y^2 = 16x$

$2y\frac{dy}{dx} = 16$ so $\frac{dy}{dx} = \frac{8}{y}$

At $(4t^2, 8t)$, $\frac{dy}{dx} = \frac{8}{8t} = \frac{1}{t}$

Gradient of tangent at $(4t^2, 8t)$ is $m_T = \frac{1}{t}$

So gradient of normal at

$(4t^2, 8t)$ is $m_N = \frac{-1}{\left(\frac{1}{t}\right)} = -t$

Normal is the line

$y - 8t = -t(x - 4t^2)$

$y - 8t = -tx + 4t^3$

$xt + y = 4t^3 + 8t$

The equation of the normal to C at

$(4t^2, 8t)$ is $xt + y = 4t^3 + 8t$

c From part **a**

P has coordinates (1, 4) when $t = \frac{1}{2}$ and

Q has coordinates (64, 32) when $t = 4$

Using equation for the normal found in part **c** (with $t = \frac{1}{2}$), normal at P is

$x\left(\frac{1}{2}\right) + y = 4\left(\frac{1}{2}\right)^3 + 8\left(\frac{1}{2}\right)$

$\frac{1}{2}x + y = \frac{1}{2} + 4$

$x + 2y = 1 + 8$

$x + 2y - 9 = 0$

Using equation for the normal found in part **c** (with $t = 4$), normal at Q is

$x(4) + y = 4(4)^3 + 8(4)$

$4x + y = 256 + 32$

$4x + y - 288 = 0$

4 d The normals to C at P and Q are $x + 2y - 9 = 0$ and $4x + y - 288 = 0$

$N_1 : x + 2y - 9 = 0$ **(1)**

$N_2 : 4x + y - 288 = 0$ **(2)**

Multiplying **(2)** by 2 gives

$2\times$**(2)**: $8x + 2y - 576 = 0$ **(3)**

(3) − **(1)**: $7x - 567 = 0$

$7x = 567$

$x = \frac{567}{7} = 81$

(2) ⇒ $y = 288 - 4(81) = 288 - 324 = -36$

The coordinates of R are $(81, -36)$

The equation of C is $y^2 = 16x$

When $y = -36$, LHS $= y^2 = (-36)^2 = 1296$

When $x = 81$, RHS $= 16x = 16(81) = 1296$

As LHS = RHS, R lies on C

e The coordinates of O and R are (0, 0) and $(81, -36)$ respectively.

$OR = \sqrt{(81-0)^2 + (-36-0)^2}$

$= \sqrt{81^2 + 36^2}$

$= \sqrt{7857}$

$= \sqrt{(81)(97)}$

$= \sqrt{81}\sqrt{97}$

$= 9\sqrt{97}$

Hence the exact distance OR is $9\sqrt{97}$ and $k = 9$

5 The focus and directrix of a parabola with equation $y^2 = 4ax$, are $(a, 0)$ and $x + a = 0$ respectively.

a Hence the coordinates of the focus of C are $(a, 0)$

As Q lies on the x-axis then $y = 0$, and since Q lies on directrix then $x = -a$ so Q has coordinates $(-a, 0)$

5 b $y^2 = 4ax$

$2y\dfrac{dy}{dx} = 4a$ so $\dfrac{dy}{dx} = \dfrac{2a}{y}$

At $P(at^2, 2at)$, $\dfrac{dy}{dx} = \dfrac{2a}{2at} = \dfrac{1}{t}$

Tangent is:

$$y - 2at = \frac{1}{t}(x - at^2)$$
$$ty - 2at^2 = x - at^2$$
$$ty = x - at^2 + 2at^2$$
$$ty = x + at^2$$

The equation of the tangent to C at P is $ty = x + at^2$ and passes through Q

Sub coordinates of Q into the equation for the tangent:

$$t(0) = -a + at^2$$
$$0 = -a + at^2$$
$$0 = -1 + t^2$$
$$t^2 - 1 = 0$$
$$(t-1)(t+1) = 0$$
$$t = 1, -1$$

When $t = 1$, $x = a(1)^2 = a$, $y = 2a(1) = 2a$
$\Rightarrow (a, 2a)$

When $t = -1$,
$x = a(-1)^2 = a$, $y = 2a(-1) = -2a$
$\Rightarrow (a, -2a)$

The possible coordinates of P are $(a,\ 2a)$ or $(a,\ -2a)$

6 a $H: xy = c^2 \Rightarrow y = c^2x^{-1}$

$\dfrac{dy}{dx} = -c^2x^{-2} = -\dfrac{c^2}{x^2}$

At $P\left(ct, \dfrac{c}{t}\right)$, $\dfrac{dy}{dx} = -\dfrac{c^2}{(ct)^2} = -\dfrac{c^2}{c^2t^2} = -\dfrac{1}{t^2}$

Gradient of tangent at $P\left(ct, \dfrac{c}{t}\right)$ is $m_T = -\dfrac{1}{t^2}$

So gradient of normal at $P\left(ct, \dfrac{c}{t}\right)$ is $m_T = \dfrac{-1}{\left(-\frac{1}{t^2}\right)} = t^2$

Normal is the line:

$$y - \frac{c}{t} = t^2(x - ct)$$
$$ty - c = t^3(x - ct)$$
$$ty - c = t^3x - ct^4$$
$$ct^4 - c = t^3x - ty$$
$$t^3x - ty = ct^4 - c$$
$$t^3x - ty = c(t^4 - 1)$$

The equation of the normal to H at P is $t^3x - ty = c(t^4 - 1)$

b Comparing $xy = 36$ with $xy = c^2$ gives $c = 6$ and comparing the point $(12, 3)$ with $\left(ct, \dfrac{c}{t}\right)$ gives

$ct = 12 \Rightarrow (6)t = 12 \Rightarrow t = 2.$

So n is the line

$$(2)^3x - (2)y = 6((-2)^4 - 1)$$
$$8x - 2y = 6(15)$$
$$8x - 2y = 90$$
$$4x - y = 45$$

An equation for n is $4x - y = 45$

6 c Normal n: $4x - y = 45$ **(1)**

Hyperbola J: $xy = 36$ **(2)**

Rearranging **(2)** gives

$$y = \frac{36}{x}$$

Substituting this equation into **(1)** gives

$$4x - \left(\frac{36}{x}\right) = 45$$

$$4x^2 - 36 = 45x$$

$$4x^2 - 45x - 36 = 0$$

$$(x - 12)(4x + 3) = 0$$

$$x = 12, -\frac{3}{4}$$

You already known that n passes through the point where $x = 12$

So at Q, $x = -\frac{3}{4}$

Substituting $x = -\frac{3}{4}$ into $y = \frac{36}{x}$ gives

$$y = \frac{36}{\left(-\frac{3}{4}\right)} = -36\left(\frac{4}{3}\right) = -48$$

Hence the coordinates of Q are

$$\left(-\frac{3}{4}, -48\right)$$

7 $H : xy = 9 \Rightarrow y = 9x^{-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -9x^{-2} = -\frac{9}{x^2}$$

Gradients of tangent lines l_1 and l_2 are both $-\frac{1}{4}$ implies

$$-\frac{9}{x^2} = -\frac{1}{4}$$

$$\Rightarrow x^2 = 36$$

$$\Rightarrow x = \pm\sqrt{36}$$

$$\Rightarrow x = \pm 6$$

When $x = 6$, $6y = 9 \Rightarrow y = \frac{9}{6} = \frac{3}{2}$

Let l_1 be the tangent to C at $\left(6, \frac{3}{2}\right)$

When $x = -6$, $-6y = 9 \Rightarrow y = \frac{9}{-6} = -\frac{3}{2}$

Let l_2 be the tangent to C at $\left(-6, -\frac{3}{2}\right)$

At $\left(6, \frac{3}{2}\right)$, $m_T = -\frac{1}{4}$ and l_1 is the line

$$y - \frac{3}{2} = -\frac{1}{4}(x - 6)$$

$$4y - 6 = -1(x - 6)$$

$$4y - 6 = -x + 6$$

$$x + 4y - 12 = 0$$

At $\left(-6, -\frac{3}{2}\right)$, $m_T = -\frac{1}{4}$ and l_2 is the line

$$y + \frac{3}{2} = -\frac{1}{4}(x + 6)$$

$$4y + 6 = -1(x + 6)$$

$$4y + 6 = -x - 6$$

$$x + 4y + 12 = 0$$

The equation for l_1 and l_2 are

$x + 4y - 12 = 0$ and

$x + 4y + 12 = 0$

8 a $H: xy = c^2 \Rightarrow y = c^2x^{-1}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -c^2x^{-2} = -\frac{c^2}{x^2}$

At $P\left(ct, \frac{c}{t}\right), \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{c^2}{(ct^2)} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

So tangent at P has equation:

$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$t^2y - ct = -(x - ct)$

$t^2y - ct = -x + ct$

$x + t^2y = 2ct$

Tangent cuts x-axis

$\Rightarrow y = 0 \Rightarrow x + t^2(0) = 2ct \Rightarrow x = 2ct$

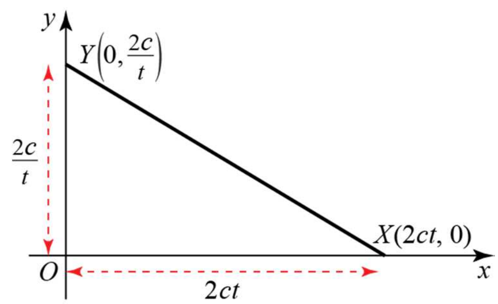
Tangent cuts y-axis

$\Rightarrow x = 0 \Rightarrow 0 + t^2y = 2ct \Rightarrow y = \frac{2ct}{t^2} = \frac{2c}{t}$

So the coordinates are

$X(2ct, 0)$ and $Y\left(0, \frac{2c}{t}\right)$

b



Using the sketch,

Area $\Delta OXY = \frac{1}{2}(2ct)\left(\frac{2c}{t}\right) = \frac{4c^2t}{2t} = 2c^2$

Now $\Delta OXY = 144$

so $2c^2 = 144 \Rightarrow c^2 = 72$

As $c > 0, c = \sqrt{72} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$

Hence the exact value of c is $6\sqrt{2}$

9 a $y^2 = 4ax$

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$ so $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{y}$

At $P(4at^2, 4at)$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{4at} = \frac{1}{2t}$

Tangent is:

$y - 4at = \frac{1}{2t}(x - 4at^2)$

$2ty - 8at^2 = x - 4at^2$

$2ty = x - 4at^2 + 8at^2$

$2ty = x + 4at^2$

The equation of the tangent to C at $P(4at^2, 4at)$ is $2ty = x + 4at^2$

b P has mapped onto Q by replacing t by $2t$, i.e. $t \to 2t$

So, $P(4at^2, 4at) \to$

$Q(16at^2, 8at) = Q(4a(2t)^2, 4a(2t))$

At Q, tangent becomes

$2(2t)y = x + 4a(2t)^2$

$4ty = x + 4a(4t^2)$

$4ty = x + 16at^2$

The equation of the tangent to C at $Q(16at^2, 8at)$ is $4ty = x + 16at^2$

c $\mathrm{T}_P : 2ty = x + 4at^2$ **(1)**

$\mathrm{T}_Q : 4ty = x + 16at^2$ **(2)**

(2) – **(1)** gives $2ty = 12at^2$

Hence, $y = \frac{12at^2}{2t} = 6at$

Substituting this into **(1)** gives,

$2t(6at) = x + 4at^2$

$12at^2 = x + 4at^2$

$12at^2 - 4at^2 = x$

Hence, $x = 8at^2$

The coordinates of R are $(8at^2, 6at)$.

10 a $H: xy = c^2 \Rightarrow y = c^2x^{-1}$

$$\frac{dy}{dx} = -c^2x^{-2} = -\frac{c^2}{x^2}$$

At $\left(ct, \frac{c}{t}\right)$, $\frac{dy}{dx} = -\frac{c^2}{(ct)^2} = -\frac{c^2}{c^2t^2} = -\frac{1}{t^2}$

Tangent is the line:

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$t^2y - ct = -(x - ct)$$

$$t^2y - ct = -x + ct$$

$$x + t^2y = 2ct$$

An equation a tangent to H at $\left(ct, \frac{c}{t}\right)$, is $x + t^2y = 2ct$

b Let T be the tangent to H at P.

T passes through $X(2a, 0)$, so substitute $x = 2a, y = 0$ into equation of a general tangent (found in part **a**) to find the value of t for T:

$$(2a) + t^2(0) = 2ct$$

$$\frac{2a}{2c} = t$$

$$t = \frac{a}{c}$$

Now P lies on H, so its coordinates are of the form $\left(ct, \frac{c}{t}\right)$ where $t = \frac{a}{c}$

Substituting $t = \frac{a}{c}$ into $\left(ct, \frac{c}{t}\right)$ gives

$$P\left(c\left(\frac{a}{c}\right), \frac{c}{\left(\frac{a}{c}\right)}\right) = P\left(a, \frac{c^2}{a}\right)$$

Hence P has coordinates $P\left(a, \frac{c^2}{a}\right)$

10 c Substituting $x = 2a$ into the curve H gives

$$(2a)y = c^2 \Rightarrow y = \frac{c^2}{2a}$$

The y-coordinate of Q is $y = \frac{c^2}{2a}$

d The coordinates of O and Q are $(0, 0)$ and $\left(2a, \frac{c^2}{2a}\right)$

$$m_{OQ} = \frac{\frac{c^2}{2a} - 0}{2a - 0} = \frac{c^2}{2a(2a)} = \frac{c^2}{4a^2}$$

$$OQ: y - 0 = \frac{c^2}{4a^2}(x - 0)$$

$$OQ: y = \frac{c^2x}{4a^2} \quad \textbf{(1)}$$

The equation of OQ is $y = \frac{c^2x}{4a^2}$

10 e The coordinates of X and P are $(2a, 0)$ and $\left(a, \frac{c^2}{a}\right)$

$$m_{XP} = \frac{\frac{c^2}{a} - 0}{a - 2a} = \frac{\frac{c^2}{a}}{-a} = -\frac{c^2}{a^2}$$

$$XP: y - 0 = -\frac{c^2}{a^2}(x - 2a)$$

$$XP: y = -\frac{c^2}{a^2}(x - 2a) \quad \textbf{(2)}$$

Equating **(1)** and **(2)** gives

$$\frac{c^2x}{4a^2} = -\frac{c^2}{a^2}(x - 2a)$$

$$\frac{x}{4} = -(x - 2a)$$

$$\frac{x}{4} = -x + 2a$$

$$\frac{5x}{4} = 2a$$

$$x = \frac{4(2a)}{5} = \frac{8a}{5}$$

The x-coordinate of R is $\frac{8a}{5}$

10 f From part **d**, $m_{OQ} = \dfrac{c^2}{4a^2}$ and

from part **e**, $m_{XP} = -\dfrac{c^2}{a^2}$

OQ is perpendicular to *XP*, so

$$-1 = m_{OQ} \times m_{XP}$$
$$-1 = \left(\frac{c^2}{4a^2}\right)\left(-\frac{c^2}{a^2}\right)$$
$$-1 = \frac{-c^4}{4a^4}$$
$$-4a^4 = -c^4$$
$$c^4 = 4a^4$$
$$(c^2)^2 = 4a^4$$
$$c^2 = \sqrt{4a^4} = \sqrt{4}\sqrt{a^4} = 2a^2$$

Hence, $c^2 = 2a^2$, as required.

g From part **e,** At $R, x = \dfrac{8a}{5}$

Substituting

$x = \dfrac{8a}{5}$ into $y = \dfrac{c^2 x}{4a^2}$ gives,

$$y = \frac{c^2}{4a^2}\left(\frac{8a}{5}\right) = \frac{8ac^2}{20a^2}$$

and using the $c^2 = 2a^2$ from **f** gives,

$$y = \frac{8a(2a^2)}{20a^2} = \frac{16a^3}{20a^2} = \frac{4a}{5}$$

The *y*-coordinate of *R* is $\dfrac{4a}{5}$

11 a $2x - y - 12 = 0$, so $y = 2x - 12$

Solving $y = 2x - 12$ and $y^2 = 12x$ simultaneously gives:

$$(2x-12)^2 = 12x$$
$$4x^2 - 48x + 144 = 12x$$
$$4x^2 - 60x + 144 = 0$$
$$x^2 - 15x + 36 = 0$$
$$(x-3)(x-12) = 0$$

So $x = 3 \Rightarrow y^2 = 36 \Rightarrow y = -6$ (point *P*)

and $x = 12 \Rightarrow y^2 = 144 \Rightarrow y = 12$ (point *Q*)

Therefore the coordinates are $P(3,-6)$ and $Q(12,12)$

b The shaded area *R* is given by

$$\left|\int_3^{12} -\sqrt{12}x^{\frac{1}{2}}\,dx\right| - (12-3)\times 6$$
$$= \left|\left[-\frac{2\sqrt{12}x^{\frac{3}{2}}}{3}\right]_3^{12}\right| - 54$$
$$= \left|-\frac{2\sqrt{12}\left(\sqrt{12}\right)^3}{3} + \frac{2\sqrt{12}\left(\sqrt{3}\right)^3}{3}\right| - 54$$
$$= \left|-\frac{288}{3} + \frac{36}{3}\right| - 54$$
$$= \frac{288}{3} - \frac{36}{3} - \frac{162}{3}$$
$$= \frac{90}{3} = 30$$

12 a $y^2 = 36x$

$2y\frac{dy}{dx} = 36$

$\frac{dy}{dx} = \frac{18}{y}$

At P, $y = 18p \Rightarrow \frac{dy}{dx} = \frac{18}{18p} = \frac{1}{p}$

The gradient of the normal at P is therefore $-p$

The equation for l is therefore

$\frac{y-18p}{x-9p^2} = -p$

$y - 18p = -p\left(x - 9p^2\right)$

$y - 18p = -px + 9p^3$

$y + px = 18p + 9p^3$ **(1)**

b At $T(27,0)$,

$0 + 27p = 18p + 9p^3$

$9p^3 - 9p = 0$

$9p\left(p^2 - 1\right) = 0$

$9p(p+1)(p-1) = 0$

$p = 0 \Rightarrow P(0,0)$

$p = 1 \Rightarrow P(9,18)$

$p = -1 \Rightarrow P(9,-18)$

12 c l has positive gradient, so P has coordinates $P(9,-18)$

So $p = -1$

Substituting $p = -1$ into equation **(1)** gives

$y - x = -18 - 9$

$y = x - 27$

Now solving $y = x - 27$ and $y^2 = 36x$ simultaneously:

$(x-27)^2 = 36x$

$x^2 - 54x + 729 = 36x$

$x^2 - 90x + 729 = 0$

$(x-9)(x-81) = 0$

$x \neq 9$ (since $x = 9$ at P), so $x = 81$ and $y = 81 - 27 = 54$

The coordinates of Q are therefore $Q(81,54)$

d The shaded region is given by

$\int_0^{81} 6x^{\frac{1}{2}}\,dx - \frac{1}{2}\times(81-27)\times 54$

$= \left[4x^{\frac{3}{2}}\right]_0^{81} - 1458$

$= 4\times 9^3 - 1458$

$= 2916 - 1458$

$= 1458$

13 a The gradient of PQ is given by

$\frac{2aq - 2ap}{aq^2 - ap^2} = \frac{2q - 2p}{q^2 - p^2}$

$= \frac{2(q-p)}{(q+p)(q-p)} = \frac{2}{p+q}$

The equation of the line joining P and Q is therefore:

$\frac{y - 2ap}{x - ap^2} = \frac{2}{p+q}$

$(p+q)(y - 2ap) = 2\left(x - ap^2\right)$

$(p+q)y - 2ap(p+q) = 2x - 2ap^2$

$(p+q)y - 2ap^2 - 2apq = 2x - 2ap^2$

$(p+q)y - 2apq = 2x$

$(p+q)y - 2x = 2apq$

b The line PQ passes through the focus $S(a,0)$, so $0 - 2a = 2apq$

Therefore $pq = -1$

13 c $y^2 = 4ax$

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4a$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4a}{2y} = \frac{2a}{y}$

At P, $y = 2ap \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2a}{2ap} = \frac{1}{p}$

The equation of the tangent at point P is therefore $\frac{y-2ap}{x-ap^2} = \frac{1}{p}$,

or $y = \frac{x-ap^2}{p} + 2ap$ **(1)**

Similarly, the equation of the tangent at point Q is

$y = \frac{x-aq^2}{q} + 2aq$ **(2)**

Solving equations **(1)** and **(2)** simultaneously:

$\frac{x-aq^2}{q} + 2aq = \frac{x-ap^2}{p} + 2ap$

$p(x-aq^2) + 2apq^2 = q(x-ap^2) + 2ap^2q$

$px - apq^2 = qx - aqp^2 + 2ap^2q - 2apq^2$

$px - qx = apq^2 - aqp^2 + 2ap^2q - 2apq^2$

$px - qx = ap^2q - apq^2$

$x(p-q) = apq(p-q)$

$p \neq q$, so $x = apq$

Substituting in equation (1):

$y = \frac{apq - ap^2}{p} + 2ap$

$y = aq - ap + 2ap$

$y = ap + aq$

$y = a(p+q)$

The point of intersection is therefore $(apq, a(p+q))$

d Since $pq = -1$, $x = apq = -a$, which is the equation of the directrix.

14 a Suppose $P\left(ct, \frac{c}{t}\right)$ is a general point on the hyperbola $xy = c^2$, or $y = \frac{c^2}{x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \frac{\frac{-c}{t^2}}{c} = -\frac{1}{t^2}$

The tangent to the hyperbola at P has equation

$\frac{y - \frac{c}{t}}{x - ct} = -\frac{1}{t^2}$

$t^2\left(y - \frac{c}{t}\right) = -(x - ct)$

$t^2y - ct = -x + ct$

$x + t^2y = 2ct$

At A, $y = 0$, so $x = 2ct$ and the coordinates of A are $A(2ct, 0)$

At B, $x = 0$, so $t^2y = 2ct$ and $y = \frac{2c}{t}$

The coordinates of B are therefore $B\left(0, \frac{2c}{t}\right)$

$AP^2 = (ct - 2ct)^2 + \left(\frac{c}{t} - 0\right)^2 = c^2t^2 + \frac{c^2}{t^2}$

$BP^2 = (ct - 0)^2 + \left(\frac{c}{t} - \frac{2c}{t}\right)^2 = c^2t^2 + \frac{c^2}{t^2}$

$AP^2 = BP^2$, so $AP = BP$

b The area of triangle AOB is

$\frac{1}{2} \times 2ct \times \frac{2c}{t} = 2c^2$,

so the area is constant as it does not depend on t.

15 a The gradient of PQ is given by

$$\frac{2aq-2ap}{aq^2-ap^2}=\frac{2q-2p}{q^2-p^2}$$
$$=\frac{2(q-p)}{(q+p)(q-p)}=\frac{2}{p+q}$$

The equation of the line joining P and Q is therefore:

$$\frac{y-2ap}{x-ap^2}=\frac{2}{p+q}$$
$$(p+q)(y-2ap)=2(x-ap^2)$$
$$(p+q)y-2ap(p+q)=2x-2ap^2$$
$$(p+q)y-2ap^2-2apq=2x-2ap^2$$
$$(p+q)y-2apq=2x$$
$$(p+q)y-2x=2apq$$

The line PQ passes through the focus $S(a,0)$, so $0-2a=2apq$

Therefore $pq=-1$

$$y^2=4ax$$
$$2y\frac{\mathrm{d}y}{\mathrm{d}x}=4a$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4a}{2y}=\frac{2a}{y}$$

At $y=2ap$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2a}{2ap}=\frac{1}{p}$

The equation of the tangent at point P is therefore $\frac{y-2ap}{x-ap^2}=\frac{1}{p}$,

or $y=\frac{x-ap^2}{p}+2ap$ **(1)**

Similarly, the equation of the tangent at point Q is

$y=\frac{x-aq^2}{q}+2aq$ **(2)**

Solving equations **(1)** and **(2)** simultaneously:

$$\frac{x-aq^2}{q}+2aq=\frac{x-ap^2}{p}+2ap$$
$$p(x-aq^2)+2apq^2=q(x-ap^2)+2ap^2q$$
$$px-apq^2=qx-aqp^2+2ap^2q-2apq^2$$
$$px-qx=apq^2-aqp^2+2ap^2q-2apq^2$$
$$px-qx=ap^2q-apq^2$$
$$x(p-q)=apq(p-q)$$

$p\neq q$, so $x=apq$

Since $pq=-1$, $x=apq=-a$,
which is the equation of the directrix.

Therefore the two tangents meet on the directrix.

15 b The midpoint M of PQ has coordinate

$$M\left(\frac{ap^2+aq^2}{2},\frac{2ap+2aq}{2}\right)$$
$$=M\left(\frac{a(p^2+q^2)}{2},a(p+q)\right)$$

Now $y^2=a^2(p+q)^2$

and

$$2a(x-a)=2a\left(\frac{a(p^2+q^2)}{2}-a\right)$$
$$=2a^2\left(\frac{p^2+q^2}{2}-1\right)$$
$$=2a^2\left(\frac{(p+q)^2-2pq}{2}-1\right)$$
$$=2a^2\left(\frac{(p+q)^2+2}{2}-1\right)$$

since $pq=-1$

$$=2a^2\left(\frac{(p+q)^2}{2}+1-1\right)$$
$$=2a^2\left(\frac{(p+q)^2}{2}\right)$$
$$=a^2(p+q)^2=y^2$$

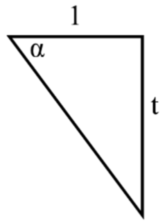
Therefore the locus of the midpoint of PQ has equation $y^2=2a(x-a)$

Challenge

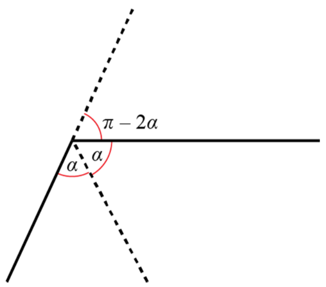
a For a general point $P\left(at^2, 2at\right)$ on the parabola, $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2a}{2at} = \frac{1}{t}$

The gradient of the normal at P is therefore $-t$

Since the incoming ray is parallel to the x-axis, from the following right-angled triangle, you have that $\tan\alpha = t$.



b $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2t}{1-t^2}$



From the diagram above, the gradient of the reflected ray is

$$\tan(\pi - 2\alpha) = -\tan 2\alpha$$
$$= -\frac{2t}{1-t^2}$$
$$= \frac{2t}{t^2-1}$$

c The focus of the parabola has coordinates $S(a, 0)$

The gradient of PS is therefore

$$\frac{2at-0}{at^2-a} = \frac{2t}{t^2-1}$$

The reflected ray has a common point (P) and the same gradient as PS. The ray therefore passes through the point S.