Recurrence relations 4A

1 a Let u_n be the value of the endowment policy after n years.
During a year, the policy increase is $0.05u_{n-1}$.
So $u_n = u_{n-1} + 0.05u_{n-1} = 1.05u_{n-1}$, $u_0 = 7000$.

b $u_1 = 1.05u_0 = 1.05 \times 7000 = 7350$
$u_2 = 1.05u_1 = 1.05 \times 7350 = 7717.5$
$u_3 = 1.05u_2 = 1.05 \times 7717.5 = 8103.375$
$u_4 = 1.05u_3 = 1.05 \times 8103.375 = 8508.54375$
So the value of the endowment after 4 years is £8508.54.

2 a During an eight hour period, $0.78\,d_{n-1}$ of the drug remains and 25 ml is added.
So $d_n = 0.78d_{n-1} + 25$, $d_0 = 156$

b After 24 hours, $n = 3$
$d_1 = 0.78d_0 + 25 = 0.78 \times 156 + 25 = 146.68$
$d_2 = 0.78d_1 + 25 = 0.78 \times 146.68 + 25 = 139.4104$
$d_3 = 0.78d_2 + 25 = 0.78 \times 139.4104 + 25 = 133.740112$
Hence, to the nearest ml, the amount of drug in the patient's system is 134 ml.

3 Each month 0.5% is added to the balance and £200 paid off so
$b_n = b_{n-1} + 0.005b_{n-1} - 200 = 1.005b_{n-1} - 200$ with $b_0 = 5000$
Hence $k = 1.005$

4 The percentage change in population due to births and deaths is +1% and the net migration into the country is 50 000.
So $P_n = P_{n-1} + 0.01P_{n-1} + 50\,000 = P_n = 1.01P_{n-1} + 50\,000$ with $P_0 = 12\,500\,000$

5 $u_{n-1} = 5(n-1) + 2 = 5n - 3$
So $u_{n-1} + 5 = 5n - 3 + 5 = 5n + 2 = u_n$

6 $u_{n-1} = 6 \times 2^{n-1} + 1$
So $2u_{n-1} - 1 = 2\left(6 \times 2^{n-1} + 1\right) - 1 = 6 \times 2 \times 2^{n-1} + 2 - 1 = 6 \times 2^n + 1 = u_n$

7 a Substituting $i = 1, 2, 3, 4$ gives 1, 3, 5, 7
The sequence is formed by summing terms so
$u_1 = 1$
$u_2 = 1 + 3 = 4$
$u_3 = 1 + 3 + 5 = 9$
$u_4 = 1 + 3 + 5 + 7 = 16$

b $u_{n+1} = \sum_{i=1}^{n} (2i - 1) + 2(n+1) - 1 = u_n + 2n + 1$, $n \geqslant 1$

7 c $u_{n+1} = (n+1)^2 = n^2 + 2n + 1 = u_n + 2n + 1$

8 a i The increase in oil production every month is 1%, so the multiplier is 1.01.
Hence after n months, the amount of oil produced is $2000 \times 1.01^{n-1}$.

ii The increase in sales every month is 20.
Hence after n months, the amount of barrels sold is $1800 + 20(n - 1) = 1780 + 20n$.

b $s_n = s_{n-1} + 2000 \times 1.01^{n-1} - (1780 + 20n) = s_{n-1} + 2000 \times 1.01^{n-1} - 1780 - 20n$ with $s_0 = 0$

9 a With one person, there is no one else to shake hands with so there are no handshakes.

b Each person added shakes hands with the existing n people in the group so
$h(n + 1) = h(n) + n$.

10 a $u_0 = 1, u_1 = 1$ as given
$u_2 = 2\times1 + 3\times1 = 5$
$u_3 = 2\times5 + 3\times1 = 13$
$u_4 = 2\times13 + 3\times5 = 41$
$u_5 = 2\times41 + 3\times13 = 121$

b $u_0 = 1, u_1 = 1$ as given
$u_2 = 1 - 2\times1 = -1$
$u_3 = -1 - 2\times1 = -3$
$u_4 = -3 - 2\times-1 = -1$
$u_5 = -1 - 2\times-3 = 5$

c $u_0 = 1, u_1 = 1$ as given
$u_2 = 1 + 1 + 2\times2 = 6$
$u_3 = 6 + 1 + 2\times3 = 13$
$u_4 = 13 + 6 + 2\times4 = 27$
$u_5 = 27 + 13 + 2\times5 = 50$

11 After n hours, bacteria alive after $n - 1$ hours double, so $+2B_{n-1}$ while bacteria alive after $n - 2$ hours die so $-B_{n-2}$.
Hence after n hours, the number of bacteria is $B_n = 2B_{n-1} - B_{n-2}$, $n \geqslant 2$ with $B_0 = 100$

12 $u_{n-1} = (2 - (n-1))2^{n-1+1} = (3-n)2^n$
$u_{n-2} = (2 - (n-2))2^{n-2+1} = (4-n)2^{n-1}$

$$
\begin{aligned}
4(u_{n-1} - u_{n-2}) &= 2^2(3-n)2^n - 2^2(4-n)2^{n-1} \\
&= 2(3-n)2^{n+1} - (4-n)2^{n+1} \\
&= 2^{n+1}(2(3-n) - (4-n)) \\
&= (2-n)2^{n+1}
\end{aligned}
$$

which equals u_n as required.

13 a The options are: 10, 10, 10, 10; 10, 10, 20; 10, 20, 10; 20, 10, 10; 20, 20
Hence there are 5 options and $J_4 = 5$.

b $J_1 = 1$ since the kangaroo can cover 10 cm in only one way, a single small jump
$J_2 = 2$ since the kangaroo can cover 20 cm with either two small jumps or one large jump
When $n = 3$, you can either add a single large jump to the $n = 1$ options, or a single small jump to the $n = 2$ options, so there are 1 + 2 = 3 options.
Generally, when you consider J_n, you can either add one large jump to the options for J_{n-2} or a single small jump to the options for J_{n-1} hence $J_n = J_{n-1} + J_{n-2}$

c $J_5 = 5 + 3 = 8$
$J_6 = 8 + 5 = 13$
$J_7 = 13 + 8 = 21$
$J_8 = 21 + 13 = 34$

14 a Initially, there are four rabbits so $F_0 = 4$
Those four rabbits each produce six offspring so $F_1 = 6 \times 4 + 4 = 28$
In each subsequent year, $F_{n-1} - F_{n-2}$ rabbits just born produce two offspring each and the F_{n-2} older rabbits each produce six offspring.
Hence $F_n = 2(F_{n-1} - F_{n-2}) + 6F_{n-2} + F_{n-1} = 3F_{n-1} + 4F_{n-2}$ as required.

b The model assumes no female rabbits ever die.

15 a When $n = 1$, the only options are 0 and 1 so $b_1 = 2$
When $n = 2$, the options are 00, 01 and 10 so $b_2 = 3$

b Strings of length n ending with 0 that do not have consecutive 1s are the strings of length $n - 1$ with no consecutive 1s, so there are b_{n-1} such strings.
Strings of length n ending with 1 that do not have consecutive 1s must have 0 as their $(n - 1)$th digit, otherwise they will end with a pair of 1s.
It follows that the strings with length n ending with a 1 that have no consecutive 1s are the strings of length $n - 2$ with no consecutive 1s with 01 added at the end, so there are b_{n-2} such strings.
Hence $b_n = b_{n-1} + b_{n-2}$ as required.

c $b_3 = 3 + 2 = 5$
$b_4 = 5 + 3 = 8$
$b_5 = 8 + 5 = 13$
$b_6 = 13 + 8 = 21$
$b_7 = 21 + 13 = 34$