Recurrence relations 4C

1 a n even: $5(-1)^{n-1}+6(-1)^{n-2}=-5+6=1=(-1)^n$

n odd: $5(-1)^{n-1}+6(-1)^{n-2}=5-6=-1=(-1)^n$

b $5\times 6^{n-1}+6\times 6^{n-2}=5\times 6^{n-1}+6^{n-1}=6\times 6^{n-1}=6^n$

c $5(A(-1)^{n-1}+B(6^{n-1}))+6(A(-1)^{n-2}+B(6^{n-2}))$

$=-5A(-1)^{n-2}+6A(-1)^{n-2}+5B(6^{n-1})+6B(6^{n-2})$

$=A(-1)^{n-2}+6B(6^{n-1})$

$=A(-1)^n+B(6^n)$

2 a $5(3^n)-6\times 5(3^{n-1})+9\times 5(3^{n-2})$

$=5(3^n)-2\times 5(3^n)+5(3^n)=0$

b $-n3^n-6(-(n-1)3^{n-1})+9(-(n-2)3^{n-2})$

$=-n3^n+6n(3^{n-1})+6\times 3^{n-1}-9n(3^{n-2})-9\times 2(3^{n-2})$

$=-n3^n+2n3^n+2\times 3^n-n3^n-2\times 3^n=0$

c $5(3^n)-n3^n-6(5(3^{n-1})-(n-1)3^{n-1})+9(5(3^{n-2})-(n-2)3^{n-2})$

$=5(3^n)-n3^n-10(3^n)+6n3^{n-1}-6(3^{n-1})+5(3^n)-9n3^{n-2}+18(3^{n-2})$

$=5(3^n)-n3^n-10(3^n)+2n3^n-2(3^n)+5(3^n)-n3^n+2(3^n)=0$

3 a $\cos\left((n+2)\frac{\pi}{2}\right)+\cos\left(n\frac{\pi}{2}\right)=\cos\left(\pi+n\frac{\pi}{2}\right)+\cos\left(n\frac{\pi}{2}\right)$

$=-\cos\left(n\frac{\pi}{2}\right)+\cos\left(n\frac{\pi}{2}\right)=0$

b $\sin\left((n+2)\frac{\pi}{2}\right)+\sin\left(n\frac{\pi}{2}\right)=\sin\left(\pi+n\frac{\pi}{2}\right)+\sin\left(n\frac{\pi}{2}\right)$

$=-\sin\left(n\frac{\pi}{2}\right)+\sin\left(n\frac{\pi}{2}\right)=0$

c $\cos\left((n+2)\frac{\pi}{2}\right)+\sin\left((n+2)\frac{\pi}{2}\right)+\cos\left(n\frac{\pi}{2}\right)+\sin\left(n\frac{\pi}{2}\right)$

$=\cos\left(\pi+n\frac{\pi}{2}\right)+\sin\left(\pi+n\frac{\pi}{2}\right)+\cos\left(n\frac{\pi}{2}\right)+\sin\left(n\frac{\pi}{2}\right)$

$=-\cos\left(n\frac{\pi}{2}\right)+-\sin\left(n\frac{\pi}{2}\right)+\cos\left(n\frac{\pi}{2}\right)+\sin\left(n\frac{\pi}{2}\right)=0$

4 $au_{n-1}+bu_{n-2}=a(c\mathrm{F}(n-1)+d\mathrm{G}(n-1))+b(c\mathrm{F}(n-2)+d\mathrm{G}(n-2))$

$=c(a\mathrm{F}(n-1)+b\mathrm{F}(n-2))+d(a\mathrm{G}(n-1)+b\mathrm{G}(n-2))$

$=c\mathrm{F}(n)+d\mathrm{G}(n)=u_n$

5 **a** Auxiliary equation: $r^2 - 2r + 1 = 0$
Solving gives $r = 1$
General solution: $a_n = (A + Bn)(1^n) = A + Bn$

b Auxiliary equation: $r^2 - 3r + 2 = 0$
Solving gives $r = 1$ or $r = 2$
General solution: $u_n = A(1^n) + B(2^n) = A + B(2^n)$

c Auxiliary equation: $r^2 - 6r + 9 = 0$
Solving gives $r = 3$
General solution: $x_n = (A + Bn)(3^n)$

d Auxiliary equation: $r^2 - 4r + 5 = 0$
Solving gives $r = 2 \pm \mathrm{i}$
General solution: $t_n = A(2 + \mathrm{i})^n + B(2 - \mathrm{i})^n$

6 Auxiliary equation must have solutions of $r = 7$ or $r = 1$
Thus $(r - 7)(r - 1) = 0$
Hence $r^2 - 8r + 7 = 0$
Hence $a = -8$ and $b = 7$

7 **a** Auxiliary equation: $r^2 - 5r + 6 = 0$
Solving gives $r = 3$ or $r = 2$
General solution: $a_n = A(2^n) + B(3^n)$
Substituting initial conditions:
$2 = A + B$
$5 = 2A + 3B$
Solving simultaneously gives $A = 1$ and $B = 1$
Hence closed form of the recurrence relation is $a_n = 2^n + 3^n$

b Auxiliary equation: $r^2 - 6r + 9 = 0$
Solving gives $r = 3$
General solution: $u_n = (A + Bn)(3^n)$
Substituting initial conditions:
$2 = 3(A + B)$
$5 = 9(A + 2B)$
Solving simultaneously gives $A = \frac{7}{9}$ and $B = -\frac{1}{9}$
Hence closed form of the recurrence relation is $u_n = \left(\frac{7}{9} - \frac{1}{9}n\right)(3^n) = (7 - n)(3^{n-2})$

c Auxiliary equation: $r^2 - 7r + 10 = 0$
Solving gives $r = 5$ or $r = 2$
General solution: $s_n = A(2^n) + B(5^n)$
Substituting initial conditions:
$4 = A + B$
$17 = 2A + 5B$
Solving simultaneously gives $A = 1$ and $B = 3$
Hence closed form of the recurrence relation is $s_n = 2^n + 3(5^n)$

7 d Auxiliary equation: $r^2 - 2r + 5 = 0$
Solving gives $r = 1 \pm 2\text{i}$
General solution: $u_n = A(1+2\text{i})^n + B(1-2\text{i})^n$
Substituting initial conditions:
$1 = A + B$
$5 = A(1 + 2\text{i}) + B(1 - 2\text{i}) = A + B + 2(A - B)\text{i}$
Solving simultaneously gives
$2(A - B)\text{i} = 4 \Rightarrow (A - B)\text{i} = 2$
$(1 - 2B)\text{i} = 2 \Rightarrow B = \frac{\text{i}-2}{2\text{i}} = \frac{1+2\text{i}}{2}$
$\Rightarrow A = 1 - \frac{1+2\text{i}}{2} = \frac{1-2\text{i}}{2}$

Hence closed form of the recurrence relation is $u_n = \left(\frac{1-2\text{i}}{2}\right)(1+2\text{i})^n + \left(\frac{1+2\text{i}}{2}\right)(1-2\text{i})^n$

Or $u_n = \frac{5}{2}\left[(1+2\text{i})^{n-1} + (1-2\text{i})^{n-1}\right]$

8 a Auxiliary equation: $r^2 - 5r + 4 = 0$
Solving gives $r = 4$ or $r = 1$
General solution: $u_n = A(1^n) + B(4^n)$
Substituting initial conditions:
$20 = A + B$
$19 = A + 4B$
Solving simultaneously gives $A = \frac{61}{3}$ and $B = -\frac{1}{3}$

Hence closed form of the recurrence relation is $u_n = \frac{61}{3} - \frac{1}{3}(4^n)$

b $u_{n+1} - u_n = -\frac{4}{3}(4^n) - \left(-\frac{1}{3}(4^n)\right) = -4^n < 0$ so u_n is decreasing.
$u_n < 0 \Rightarrow 4^n > 61 \Rightarrow n \geqslant 3$

9 a Auxiliary equation: $r^2 - \sqrt{2}r + 1 = 0$

Solving gives $r = \frac{\sqrt{2}}{2}(1 \pm \mathrm{i})$

Complex solutions with modulus 1 and $\theta = \frac{\pi}{4}$

General solution: $u_n = 1^n\left(A\cos\left(\frac{n\pi}{4}\right) + B\sin\left(\frac{n\pi}{4}\right)\right) = A\cos\left(\frac{n\pi}{4}\right) + B\sin\left(\frac{n\pi}{4}\right)$

Substituting initial conditions:

$1 = A$

$1 = \frac{A}{\sqrt{2}} + \frac{B}{\sqrt{2}}$

Solving simultaneously gives $A = 1$ and $B = \sqrt{2}\left(1 - \frac{1}{\sqrt{2}}\right) = \sqrt{2} - 1$

Hence closed form of the recurrence relation is $u_n = \cos\left(\frac{n\pi}{4}\right) + \left(\sqrt{2} - 1\right)\sin\left(\frac{n\pi}{4}\right)$

b cos and sin are both periodic so the sequence is periodic.

Period of sin and cos is 2π so period of u_n is $\frac{2\pi}{\frac{\pi}{4}} = 8$

10 a Sequence is formed Fibonacci like so first seven terms are 1, 3, 4, 7, 11, 18, 29.

b Auxiliary equation: $r^2 - r - 1 = 0$

Solving gives $r = \frac{1 \pm \sqrt{5}}{2}$

General solution: $L_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n$

Substituting initial conditions:

$1 = A\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)$

$3 = A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = A\left(\frac{3+\sqrt{5}}{2}\right) + B\left(\frac{3-\sqrt{5}}{2}\right)$

Solving simultaneously gives $A = 1$ and $B = 1$

Hence closed form of the recurrence relation is $L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$

11 a Auxiliary equation: $r^2 - 5r + 6 = 0$
Solving gives $r = 2$ or $r = 3$
Complementary function is $x_n = A(2^n) + B(3^n)$
Try particular solution λ:

$$\lambda = 5\lambda - 6\lambda + 1 \Rightarrow \lambda = \frac{1}{2}$$

Hence general solution is $x_n = A(2^n) + B(3^n) + \frac{1}{2}$

b Auxiliary equation: $r^2 - r - 2 = 0$
Solving gives $r = -1$ or $r = 2$
Complementary function is $u_n = A(2^n) + B(-1)^n$
Try particular solution $\lambda n + \mu$:

$$\lambda n + \mu - (\lambda(n-1) + \mu) - 2(\lambda(n-2) + \mu) = 2n$$
$$\lambda n + \mu - \lambda n + \lambda - \mu - 2\lambda n + 4\lambda - 2\mu = 2n$$

Equating coefficients:

$$-2\lambda = 2 \Rightarrow \lambda = -1$$
$$\mu + \lambda - \mu + 4\lambda - 2\mu = 0 \Rightarrow -2\mu = 5 \Rightarrow \mu = -\frac{5}{2}$$

Hence general solution is $u_n = A(2^n) + B(-1)^n - n - \frac{5}{2}$

c Auxiliary equation: $r^2 + 4r + 3 = 0$
Solving gives $r = -3$ or $r = -1$
Complementary function is $a_n = A(-3)^n + B(-1)^n$
Try particular solution $\lambda(-2)^n$:

$$\lambda(-2)^{n+2} + 4\lambda(-2)^{n+1} + 3\lambda(-2)^n = 5(-2)^n$$
$$4\lambda - 8\lambda + 3\lambda = 5 \Rightarrow \lambda = -5$$

Hence general solution is $a_n = A(-3)^n + B(-1)^n - 5(-2)^n$

d Auxiliary equation: $r^2 + 4r + 3 = 0$
Solving gives $r = -3$ or $r = -1$
Complementary function is $a_n = A(-3)^n + B(-1)^n$
Try particular solution $\lambda n(-3)^n$:

$$\lambda(n+2)(-3)^{n+2} + 4\lambda(n+1)(-3)^{n+1} + 3\lambda n(-3)^n = 12(-3)^n$$
$$9\lambda(n+2) - 12\lambda(n+1) + 3\lambda n = 12$$
$$\Rightarrow 18\lambda - 12\lambda = 12 \Rightarrow \lambda = 2$$

Hence general solution is $a_n = A(-3)^n + B(-1)^n + 2n(-3)^n$

11 e Auxiliary equation: $r^2-6r+9=0$
Solving gives $r = 3$
Complementary function is $a_n = (A+Bn)(3^n)$
Try particular solution $\lambda n^2(3^n)$:
$\lambda(n+2)^2(3^{n+2})-6\lambda(n+1)^2(3^{n+1})+9\lambda n^2(3^n)=3^n$
$9\lambda(n+2)^2-18\lambda(n+1)^2+9\lambda n^2=1$
$\Rightarrow 36\lambda-18\lambda=1\Rightarrow \lambda=\frac{1}{18}$
Hence general solution is $a_n=(A+Bn)(3^n)+\frac{1}{18}n^2(3^n)=\left(A+Bn+\frac{n^2}{18}\right)(3^n)$

f Auxiliary equation: $r^2-7r+10=0$
Solving gives $r = 2$ or $r = 5$
Complementary function is $u_n = A(2^n)+B(5^n)$
Try particular solution $\lambda n+\mu$:
$\lambda n+\mu=7(\lambda(n-1)+\mu)-10(\lambda(n-2)+\mu)+6+8n$
$\lambda n+\mu=7\lambda n-7\lambda+7\mu-10\lambda n+20\lambda-10\mu+6+8n$
Equating coefficients:
$\lambda=7\lambda-10\lambda+8\Rightarrow \lambda=2$
$\mu=-7\lambda+7\mu+20\lambda-10\mu+6\Rightarrow 4\mu=32\Rightarrow \mu=8$
Hence general solution is $u_n = A(2^n)+B(5^n)+8+2n$

12 a Auxiliary equation: $r^2-2r-3=0$
Solving gives $r = 3$ or $r = -1$
Complementary function is $u_n = A(3^n)+B(-1)^n$
Try particular solution λ:
$\lambda=2\lambda+3\lambda+1\Rightarrow \lambda=-\frac{1}{4}$
Hence general solution is $u_n = A(3^n)+B(-1)^n-\frac{1}{4}$
Substituting initial conditions:
$3=3A-B-\frac{1}{4}\Rightarrow 3A-B=\frac{13}{4}$
$7=9A+B-\frac{1}{4}\Rightarrow 9A+B=\frac{29}{4}$
Solving simultaneously gives $A=\frac{7}{8}$ and $B=-\frac{5}{8}$
Hence closed form of the recurrence relation is $u_n=\frac{7}{8}(3^n)-\frac{5}{8}(-1)^n-\frac{1}{4}=\frac{1}{8}(7(3^n)-5(-1)^n-2)$

12 b Auxiliary equation: $r^2 - 3r + 2 = 0$
Solving gives $r = 2$ or $r = 1$
Complementary function is $a_n = A + B(2^n)$
Try particular solution $\lambda(-1)^n$:
$\lambda(-1)^{n+1} - 3\lambda(-1)^n + 2\lambda(-1)^{n-1} = 6(-1)^n$
$\lambda + 3\lambda + 2\lambda = -6 \Rightarrow \lambda = -1$
Hence general solution is $a_n = A + B(2^n) - (-1)^n$
Substituting initial conditions:
$12 = A + B - 1 \Rightarrow A + B = 13$
$12 = A + 2B + 1 \Rightarrow A + 2B = 11$
Solving simultaneously gives $B = -2$ and $A = 15$
Hence closed form of the recurrence relation is $a_n = 15 - 2(2^n) - (-1)^n = 15 - 2^{n+1} + (-1)^{n+1}$

c Auxiliary equation: $r^2 - 3r - 10 = 0$
Solving gives $r = 5$ or $r = -2$
Complementary function is $u_n = A(5^n) + B(-2)^n$
Try particular solution $\lambda n(5^n)$::
$\lambda n(5^n) = 3\lambda(n-1)(5^{n-1}) + 10\lambda(n-2)(5^{n-2}) + 7(5^n)$
$25\lambda n = 15\lambda(n-1) + 10\lambda(n-2) + 175$
$\Rightarrow 0 = -15\lambda - 20\lambda + 175 \Rightarrow \lambda = 5$
Hence general solution is $u_n = A(5^n) + B(-2)^n + 5n(5^n)$
Substituting initial conditions:
$4 = A + B$
$3 = 5A - 2B + 25 \Rightarrow 5A - 2B = -22$
Solving simultaneously gives $A = -2$ and $B = 6$
Hence closed form of the recurrence relation is $u_n = -2(5^n) + 6(-2)^n + 5n(5^n)$
$= n(5^{n+1}) - 2(5^n) + 6(-2)^n$

d Auxiliary equation: $r^2 - 10r + 25 = 0$
Solving gives $r = 5$
Complementary function is $x_n = (A + Bn)(5^n)$
Try particular solution $\lambda n^2(5^n)$:
$\lambda n^2(5^n) = 10\lambda(n-1)^2(5^{n-1}) - 25\lambda(n-2)^2(5^{n-2}) + 8(5^n)$
$25\lambda n^2 = 50\lambda(n-1)^2 - 25\lambda(n-2)^2 + 200$
$0 = 50\lambda - 100\lambda + 200 \Rightarrow \lambda = 4$
Hence general solution is $x_n = (A + Bn)(5^n) + 4n^2(5^n)$
Substituting initial conditions:
$6 = A$
$10 = 5A + 5B + 20 \Rightarrow B = -8$
Hence closed form of the recurrence relation is $x_n = (6 - 8n)(5^n) + 4n^2(5^n) = (4n^2 - 8n + 6)(5^n)$

13 a Try particular solution k:
$k + 4k + 4k = 7 \Rightarrow k = \frac{7}{9}$

13 b Auxiliary equation: $r^2+4r+4=0$
Solving gives $r = -2$
Complementary function is $b_n = (A+Bn)(-2)^n$

Hence general solution is $b_n = (A+Bn)(-2)^n + \frac{7}{9}$

Substituting initial conditions:

$1 = A + \frac{7}{9} \Rightarrow A = \frac{2}{9}$

$2 = -2(A+B) + \frac{7}{9} \Rightarrow B = -\frac{5}{6}$

Hence closed form of the recurrence relation is $b_n = \left(\frac{2}{9} - \frac{5}{6}n\right)(-2)^n + \frac{7}{9} = \frac{2}{9}(-2)^n - \frac{5}{6}n(-2)^n + \frac{7}{9}$

14 a Auxiliary equation: $r^2-7r+6=0$
Solving gives $r = 6$ or $r = 1$
Complementary function is $u_n = A(6^n) + B$
Try particular solution λn:

$\lambda n = 7\lambda(n-1) - 6\lambda(n-2) + 75$

$\lambda n = 7\lambda n - 7\lambda - 6\lambda n + 12\lambda + 75$

$\Rightarrow 0 = 5\lambda + 75 \Rightarrow \lambda = -15$

Hence general solution is $u_n = A(6^n) + B - 15n$

b Substituting initial conditions:

$2 = A + B$

$2 = 6A + B - 15$ $6A + B = 17$

Solving simultaneously gives $A = 3$ and $B = -1$
Hence closed form of the recurrence relation is $u_n = 3(6^n) - 1 - 15n = 3(6^n) - 15n - 1$

15 a Try particular solution $kn^2(3^n)$:

$k(n+2)^2(3^{n+2}) - 6k(n+1)^2(3^{n+1}) + 9kn^2(3^n) = 7(3^n)$

$9k(n+2)^2 - 18k(n+1)^2 + 9kn^2 = 7$

$\Rightarrow 36k - 18k = 7 \Rightarrow k = \frac{7}{18}$

b Auxiliary equation: $r^2-6r+9=0$
Solving gives $r = 3$
General solution is $u_n = (A+Bn)(3^n) = A(3^n) + Bn(3^n)$

c General solution is $u_n = A(3^n) + Bn(3^n) + \frac{7}{18}n^2(3^n)$

Substituting initial conditions:

$1 = A$

$4 = 3A + 3B + \frac{21}{18} \Rightarrow B = -\frac{1}{18}$

Hence closed form of the recurrence relation is $u_n = 3^n - \frac{1}{18}n(3^n) + \frac{7}{18}n^2(3^n) = \left(1 - \frac{1}{18}n + \frac{7}{18}n^2\right)(3^n)$

16 a Auxiliary equation: $r^2 - r + 1 = 0$

Solving gives $r = \frac{1}{2}(1 \pm \sqrt{3}\text{i})$

Complex solutions with modulus 1 and $\theta = \frac{\pi}{3}$

General solution: $u_n = 1^n\left(A\cos\left(\frac{n\pi}{3}\right) + B\sin\left(\frac{n\pi}{3}\right)\right) = A\cos\left(\frac{n\pi}{3}\right) + B\sin\left(\frac{n\pi}{3}\right)$

Substituting initial conditions:

$0 = A$

$3 = B\left(\sin\left(\frac{\pi}{3}\right)\right) \Rightarrow B = \frac{3}{\frac{\sqrt{3}}{2}} = 2\sqrt{3}$

Hence the closed form of the recurrence relation is $u_n = 2\sqrt{3}\left(\sin\left(\frac{\pi}{3}n\right)\right)$

b sin is periodic so the sequence is periodic. Period of sin = 2π so period of sequence = $\frac{2\pi}{\frac{\pi}{3}} = 6$

17 a If A is first, you cannot have A second but you can have it third so 2 × 3 options.
If A is second, you cannot have A first or third so 2 × 2 options.
If A is third, you cannot have A second but you can have it first so 3 × 2 options but ABA and ACA already counted so 4 options left.
Having no As gives 2 × 2 × 2 options.
6 + 4 + 4 + 8 = 22

b Strings of length n with no consecutive As are the strings of length $n - 1$ with no consecutive As with a B or a C added at the end.

Thus there are $2s_{n-1}$ such strings.

But strings of length n ending in an A that do not have consecutive As must have B or C as their $(n - 1)$th letter, otherwise they will end in a pair of As. It follows that strings of length n ending with an A that have no consecutive As are the strings of length $n - 2$ with either a B or a C added at the end.

Thus there are $2s_{n-2}$ such strings.

Hence $s_n = 2s_{n-1} + 2s_{n-2}$ with $s_0 = 1$ and $s_1 = 3$

c i Auxiliary equation: $r^2 - 2r - 2 = 0$

Solving gives $r = 1 \pm \sqrt{3}$

General solution is $s_n = A(1+\sqrt{3})^n + B(1-\sqrt{3})^n$

Substituting initial conditions:

$1 = A + B$

$3 = A(1+\sqrt{3}) + B(1-\sqrt{3})$

Solving simultaneously gives $A = \frac{3+2\sqrt{3}}{6}$ and $B = \frac{3-2\sqrt{3}}{6}$

Hence closed form of the recurrence relation is $s_n = \frac{1}{6}\left[(3+2\sqrt{3})(1+\sqrt{3})^n + (3-2\sqrt{3})(1-\sqrt{3})^n\right]$

ii Substitute $n = 20$: $s_{20} = 578272256$

Challenge

1 $u_n = \sqrt{\frac{u_{n-2}}{u_{n-1}}}, \quad u_0 = 8, \quad u_1 = \frac{1}{2\sqrt{2}}$

$\log_2 u_n = \frac{1}{2}\log_2 u_{n-2} - \frac{1}{2}\log_2 u_{n-1}$

Let $v_n = \log_2 u_n$

Then $v_0 = \log_2 8 \qquad v_1 = \log_2\left(2^{-\frac{3}{2}}\right) = -\frac{3}{2}$

The recurrence relation becomes $v_n = \frac{1}{2}v_{n-2} - \frac{1}{2}v_{n-1}$

The auxiliary equation is

$r^2 + \frac{1}{2}r - \frac{1}{2} = 0$

$2r^2 + r - 1 = 0$

$(2r-1)(r+1) = 0$

$r = -1$ or $\frac{1}{2}$

So the general solution has the form

$v_n = A(-1)^n + B\left(\frac{1}{2}\right)^n$

$$\left.\begin{array}{ll} n=0: & A+B=3 \\ n=1: & -A+0.5B=3 \end{array}\right\} \frac{3}{2}B = \frac{3}{2} \Rightarrow B=1,\ A=2$$

$v_n = 2(-1)^n + \left(\frac{1}{2}\right)^n$

and $u_n = 2^{v_n} = 2^{2(-1)^n + \left(\frac{1}{2}\right)^n}$

2 Need auxiliary equation to have complex solutions such that $\theta = \frac{\pi}{6}$, i.e. complex solutions of the form $p \pm q\mathrm{i}$ such that $\frac{q}{p} = \frac{1}{\sqrt{3}}$ hence complex solutions $\sqrt{3} \pm \mathrm{i}$.

$(r - (\sqrt{3} + \mathrm{i}))(r - (\sqrt{3} - \mathrm{i})) = 0$

$\Rightarrow r^2 - 2\sqrt{3}r + 4 = 0$

Hence possible recurrence relation is $u_n = 2\sqrt{3}u_{n-1} - 4u_{n-2}$ giving $a = 2\sqrt{3}$ and $b = -4$

Magnitude of complex solutions is 2 so $u_n = 2\left(A\cos\left(n\frac{\pi}{6}\right) + B\sin\left(n\frac{\pi}{6}\right)\right)$

Using initial conditions:

$0 = 2A \Rightarrow A = 0$

$k = 2\left(A\cos\left(\frac{\pi}{6}\right) + B\sin\left(\frac{\pi}{6}\right)\right) \Rightarrow k = B$

Therefore particular solution is $u_n = 2k\sin\left(n\frac{\pi}{6}\right)$ so sequence is periodic with period 12 and oscillates between $\pm 2k$