Integration techniques 6B

1 $y=\frac{1}{3}x^{\frac{3}{2}}$, so $\frac{dy}{dx}=\frac{1}{2}x^{\frac{1}{2}}$

$$\text{Arc length} = \int_0^{12}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^{12}\sqrt{1+\frac{x}{4}}\,dx$$

$$=\frac{1}{2}\int_0^{12}\sqrt{4+x}\,dx$$

$$=\frac{1}{2}\left[\frac{2}{3}(4+x)^{\frac{3}{2}}\right]_0^{12}$$

$$=\frac{1}{3}\left[16^{\frac{3}{2}}-4^{\frac{3}{2}}\right]$$

$$=\frac{1}{3}[64-8]$$

$$=\frac{56}{3}\text{ or } 18\ \frac{2}{3}$$

2 $y=\ln\cos x$, so $\frac{dy}{dx}=\frac{-\sin x}{\cos x}=-\tan x$

$$\text{Arc length} = \int_0^{\frac{\pi}{3}}\sqrt{1+\tan^2 x}\,dx = \int_0^{\frac{\pi}{3}}\sec x\,dx$$

$$=\left[\ln\left(\sec x+\tan x\right)\right]_0^{\frac{\pi}{3}}$$

$$=\ln\left(2+\sqrt{3}\right)$$

3 $y=2\cosh\left(\frac{x}{2}\right)$, so $\frac{dy}{dx}=\sinh\left(\frac{x}{2}\right)$

$$\text{arc length} = \int_0^{\ln 4}\sqrt{1+\sinh^2\left(\frac{x}{2}\right)}\,dx$$

$$=\int_0^{\ln 4}\cosh\left(\frac{x}{2}\right)dx$$

$$=\left[2\sinh\left(\frac{x}{2}\right)\right]_0^{\ln 4}$$

$$=2\,\frac{e^{\frac{\ln 4}{2}}-e^{-\frac{\ln 4}{2}}}{2}$$

$$=e^{\ln 2}-e^{-\ln 2}$$

As $\ln 4=\ln 2^2=2\ln 2$

$$=2-\frac{1}{2}=\frac{3}{2}$$

As $e^{\ln k}=k;\ e^{-\ln k}=e^{\ln k^{-1}}=k^{-1}$

4 $y^2 = \frac{4}{9}x^3$, so $2y\frac{dy}{dx} = \frac{4}{3}x^2 \Rightarrow \frac{dy}{dx} = \frac{2x^2}{3y} = \pm\frac{x^2}{x^{\frac{3}{2}}} = \pm\sqrt{x}$

The arc in question is above the x-axis.

$$\text{arc length} = \int_0^3 \sqrt{1+x}\,dx$$

$$= \left[\frac{2}{3}(1+x)^{\frac{3}{2}}\right]_0^3$$

$$= \frac{2}{3}[8-1] = 4\frac{2}{3}$$

5 $y = \frac{1}{2}\sinh^2 2x$, so $\frac{dy}{dx} = 2\sinh 2x\cosh 2x = \sinh 4x$

$$\text{So arc length} = \int_0^1 \sqrt{1+\sinh^2 4x}\,dx$$

$$= \int_0^1 \cosh 4x\,dx$$

$$= \frac{1}{4}[\sinh 4x]_0^1$$

$$= \frac{1}{4}\sinh 4 = 6.82 \quad (3\text{ s.f.})$$

6 $y = \frac{1}{4}\left(2x^2 - \ln x\right)$, so $\frac{dy}{dx} = x - \frac{1}{4x}$

$$1+\left(\frac{dy}{dx}\right)^2 = 1 + x^2 - \frac{1}{2} + \frac{1}{16x^2} = x^2 + \frac{1}{2} + \frac{1}{16x^2} = \left(x + \frac{1}{4x}\right)^2$$

$$\text{So arc length} = \int_1^2 \left(x + \frac{1}{4x}\right)dx$$

$$= \left[\frac{x^2}{2} + \frac{1}{4}\ln x\right]_1^2$$

$$= \left[2 + \frac{1}{4}\ln 2\right] - \left[\frac{1}{2}\right]$$

$$= \frac{1}{4}(6 + \ln 2)$$

7 We have $r = e^\theta$, $\frac{dr}{d\theta} = e^\theta$ and substitute into the equation $s = \int_\alpha^\beta \sqrt{(r)^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$ in order to get

$$s = \int_0^\pi \sqrt{e^{2\theta} + e^{2\theta}}\,d\theta$$

$$= \int_0^\pi \sqrt{2}e^\theta\,d\theta$$

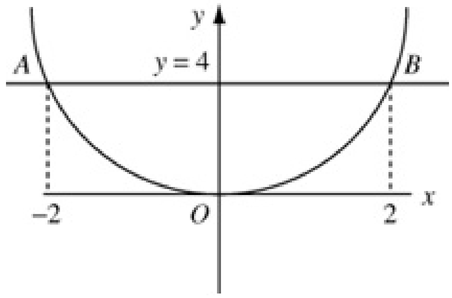
$$= \left[\sqrt{2}e^\theta\right]_0^\pi$$

$$= \sqrt{2}\left(e^\pi - 1\right)$$

8 The line $y = 4$ intersects the parabola with equation $y = x^2$ where $x = -2$ and $x = +2$.

Using symmetry arc length $= 2\int_0^2 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$

$$= 2\int_0^2 \sqrt{1+4x^2}\,dx$$



Using the substitution $2x = \sinh u$, so that $2dx = \cosh u\,du$,

$$\text{arc length} = \int_0^{\text{arsinh } 4} \sqrt{1+\sinh^2 u}\,\cosh u\,du$$

$$= \int_0^{\text{arsinh } 4} \cosh^2 u\,du$$

$$= \int_0^{\text{arsinh } 4} \frac{(1+\cosh 2u)}{2}\,du$$

$$= \frac{1}{2}\left[u + \frac{1}{2}\sinh 2u\right]_0^{\text{arsinh } 4}$$

$$= \frac{1}{2}\left[u + \sinh u \cosh u\right]_0^{\text{arsinh } 4}$$

$$= \frac{1}{2}\text{arsinh } 4 + \frac{1}{2}\left(4\sqrt{1+16}\right)$$

Using $\cosh u = \sqrt{1+\sinh^2 u}$ and $\sinh u = 4$

$$= \frac{1}{2}\text{arsinh } 4 + 2\sqrt{17}$$

$$= \frac{1}{2}\ln\left(4+\sqrt{17}\right) + 2\sqrt{17}$$

Using $\text{arsinh} x = \ln\left\{x + \sqrt{\left(1+x^2\right)}\right\}$

$$= 9.29\,(3\text{ s.f.})$$

9 As $x = a\cos\theta, y = a\sin\theta, \frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = a\cos\theta$

So $\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\left(\cos^2\theta + \sin^2\theta\right) = a^2$

The circumference of the circle $= 4\int_0^{\frac{\pi}{2}} a\,d\theta$

Using symmetry.

$$= 4a\left[\theta\right]_0^{\frac{\pi}{2}}$$

$$= 2\pi a$$

10 $x = 2a\cos^3 t, y = 2a\sin^3 t$, so $\frac{dx}{dt} = -6a\cos^2 t\sin t, \frac{dy}{dt} = 6a\sin^2 t\cos t$,

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 36a^2\left(\cos^4 t\sin^2 t + \sin^4 t\cos^2 t\right) = 36a^2\sin^2 t\cos^2 t\left(\cos^2 t + \sin^2 t\right)$$

$$= 36a^2\sin^2 t\cos^2 t$$

At $A, t = 0$, at $B, t = \frac{\pi}{2}$,

so arc length $AB = \int_0^{\frac{\pi}{2}} 6a\sin t\cos t\,dt$

$$= 3a\int_0^{\frac{\pi}{2}} \sin 2t\,dt$$

$$= \frac{3}{2}a\left[-\cos 2t\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{2}a\left[1-(-1)\right]$$

$$= 3a$$

Total length of curve $= 4 \times 3a = 12a$ (symmetry)

11 $x = \tanh u, y = \operatorname{sech} u$, so $\frac{dx}{du} = \operatorname{sech}^2 u, \frac{dy}{du} = -\operatorname{sech} u\tanh u$,

$$\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 = \operatorname{sech}^4 u + \operatorname{sech}^2 u\tanh^2 u = \operatorname{sech}^2 u\left(\operatorname{sech}^2 u + \tanh^2 u\right) = \operatorname{sech}^2 u$$

So arc length $= \int_0^1 \operatorname{sech} u\,du$ ← See Example 7.

$$= \int_0^1 \frac{2}{e^u + e^{-u}}\,du$$

$$= \int_0^1 \frac{2e^u}{\left(e^u\right)^2 + 1}\,du$$

$$= 2\left[\arctan\left(e^u\right)\right]_0^1$$

$$= 2\arctan(e) - \frac{\pi}{2} \text{ or } 0.866 \quad (3\text{ s.f.})$$

12 As $x = a(\theta + \sin\theta), y = a(1 - \cos\theta), \frac{dx}{d\theta} = a(1 + \cos\theta), \frac{dy}{d\theta} = a\sin\theta$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\left(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\right)$$

$$= a^2(2 + 2\cos\theta)$$

$$= 4a^2\cos^2\left(\frac{\theta}{2}\right)$$

Using $\cos 2A = 2\cos^2 A - 1$ with $A = \left(\frac{\theta}{2}\right)$

So arc length $= 2a\int_0^{\pi}\cos\left(\frac{\theta}{2}\right)d\theta$

$$= 4a\left[\sin\left(\frac{\theta}{2}\right)\right]_0^{\pi}$$

$$= 4a$$

13 $x = t + \sin t, y = 1 - \cos t$

$$\frac{dx}{dt} = 1 + \cos t, \frac{dy}{dt} = \sin t$$

So $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \left\{\left(1 + 2\cos t + \cos^2 t\right) + \left(\sin^2 t\right)\right\}$

$$= 2\left(1 + \cos t\right) = 4\cos^2\left(\frac{t}{2}\right)$$

Using $s = \int_{t_A}^{t_B}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$

arc length $= \int_0^{\frac{\pi}{3}}\sqrt{4\cos^2\left(\frac{t}{2}\right)}\,dt$

$$= 2\int_0^{\frac{\pi}{3}}\cos\left(\frac{t}{2}\right)dt$$

$$= 4\left[\sin\left(\frac{t}{2}\right)\right]_0^{\frac{\pi}{3}}$$

$$= 2$$

14 $x = \mathrm{e}^t \cos t, y = \mathrm{e}^t \sin t$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{e}^t(\cos t - \sin t), \frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{e}^t(\sin t + \cos t)$$

$$\text{So}\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = \left(\mathrm{e}^t\right)^2\left\{\left(\cos^2 t - 2\sin t\cos t + \sin^2 t\right) + \left(\sin^2 t + 2\sin t\cos t + \cos^2 t\right)\right\},$$

$$= 2\left(\mathrm{e}^t\right)^2\left(\sin^2 t + \cos^2 t\right)$$

$$= 2\left(\mathrm{e}^t\right)^2$$

$$\text{arc length} = \int_0^{\frac{\pi}{4}} \sqrt{2\left(\mathrm{e}^t\right)^2}\,\mathrm{d}t$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}} \mathrm{e}^t\,\mathrm{d}t$$

$$= \sqrt{2}\left[\mathrm{e}^t\right]_0^{\frac{\pi}{4}}$$

$$= \sqrt{2}\left[\mathrm{e}^{\frac{\pi}{4}} - 1\right] \text{ or } 1.69 \quad (3\text{ s.f.})$$

15 We have $r = 1 + \cos\theta$, $\frac{\mathrm{d}r}{\mathrm{d}\theta} = -\sin\theta$ and substitute into the equation $s = \int_\alpha^\beta \sqrt{(r)^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta$

in order to get

$$s = 2\int_0^\pi \sqrt{(1+\cos\theta)^2 + (-\sin\theta)^2}\,\mathrm{d}\theta$$

$$= 2\int_0^\pi \sqrt{2 + 2\cos\theta}\,\mathrm{d}\theta$$

$$= 2\sqrt{2}\int_0^\pi \sqrt{1+\cos\theta}\,\mathrm{d}\theta$$

$$= 2\sqrt{2}\int_0^\pi \sqrt{2\cos^2\left(\frac{\theta}{2}\right)}\,\mathrm{d}\theta$$

$$= 4\int_0^\pi \cos\left(\frac{\theta}{2}\right)\mathrm{d}\theta$$

$$= 4\left[2\sin\left(\frac{\theta}{2}\right)\right]_0^\pi$$

$$= 8 - 0$$

$$= 8$$

Note that we have used the fact that the cardioid has a line of symmetry through the initial line and so we may use $\int_0^{2\pi} \; = 2\int_0^\pi$

16 a We have $y=\ln(\cos x)$ so $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\sin x}{\cos x}=-\tan x$.

Now

$$\begin{aligned}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}&=\sqrt{1+(-\tan x)^2}\\&=\sqrt{\sec^2 x}\\&=\sec x\end{aligned}$$

b $s=\int_0^{\frac{\pi}{6}}\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$

$$\begin{aligned}&=\int_0^{\frac{\pi}{6}}\sec x\,\mathrm{d}x\\&=\left[\ln(\sec x+\tan x)\right]_0^{\frac{\pi}{6}}\\&=\left(\ln\left(\frac{2}{\sqrt{3}}+\frac{1}{\sqrt{3}}\right)-\ln(1)\right)\\&=\ln\left(\sqrt{3}\right)\end{aligned}$$

Thus the value of k is $\sqrt{3}$.

17 a We have $y=\ln(1-x^2)$ and so $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-2x}{1-x^2}$

Substituting into the equation $s=\int_\alpha^\beta\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$ gives

$$\begin{aligned}s&=\int_{-0.5}^{0.5}\sqrt{1+\left(\frac{-2x}{1-x^2}\right)^2}\,\mathrm{d}x\\&=\int_{-0.5}^{0.5}\frac{\sqrt{(1-x^2)^2+(2x)^2}}{1-x^2}\,\mathrm{d}x\\&=\int_{-0.5}^{0.5}\frac{\sqrt{1-2x^2+x^4+4x^2}}{1-x^2}\,\mathrm{d}x\\&=\int_{-0.5}^{0.5}\frac{1+x^2}{1-x^2}\,\mathrm{d}x\end{aligned}$$

17 b Using partial fractions we can rewrite the integral as $\frac{1+x^2}{1-x^2}=\frac{1+x^2}{(1-x)(1+x)}=\frac{1}{x+1}-\frac{1}{x-1}-1$

So we now can compute

$$\begin{aligned}
s &= \int_{-0.5}^{0.5}\frac{1+x^2}{1-x^2}\,\mathrm{d}x \\
&= \int_{-0.5}^{0.5}\left(\frac{1}{x+1}-\frac{1}{x-1}-1\right)\mathrm{d}x \\
&= \left[\ln|x+1|-\ln|x-1|-x\right]_{-0.5}^{0.5} \\
&= \left(\ln|1.5|-\ln|0.5|-0.5\right)-\left(\ln|0.5|-\ln|-1.5|+0.5\right) \\
&= 2\ln|1.5|-2\ln|0.5|-1 \\
&= 2\ln 3-2\ln 2+2\ln 2-1 \\
&= 2\ln 3-1
\end{aligned}$$

18 First we find the length of the segment to the left of the origin $x=-y^{\frac{3}{2}}$ and so $\frac{\mathrm{d}x}{\mathrm{d}y}=\frac{-3\sqrt{y}}{2}$

We use the equation

$$\begin{aligned}
s &= \int_0^1\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2+1}\,\mathrm{d}y \\
&= \int_0^1\sqrt{\frac{9y}{4}+1}\,\mathrm{d}y \\
&= \left[\frac{8}{27}\left(\frac{9y}{4}+1\right)^{\frac{3}{2}}\right]_0^1 \\
&= \frac{8}{27}\left(\left(\frac{13}{4}\right)^{\frac{3}{2}}-1\right)
\end{aligned}$$

18 (continued)

Now we find the length of the segment to the right of the origin $x = y^{\frac{3}{2}}$ and so $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{3\sqrt{y}}{2}$

We use the equation

$$s = \int_0^4 \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2 + 1}\,\mathrm{d}y$$

$$= \int_0^4 \sqrt{\frac{9y}{4} + 1}\,\mathrm{d}y$$

$$= \left[\frac{8}{27}\left(\frac{9y}{4} + 1\right)^{\frac{3}{2}}\right]_0^4$$

$$= \frac{8}{27}\left((10)^{\frac{3}{2}} - 1\right)$$

Adding the above two results gives $s = \frac{8}{27}\left(\left(\frac{13}{4}\right)^{\frac{3}{2}} + 10^{\frac{3}{2}} - 2\right)$

$$\approx 10.5 \text{ cm.}$$

19 a Since we have $y = a\cosh\left(\frac{x}{a}\right) - 40$, we have $\frac{\mathrm{d}y}{\mathrm{d}x} = \sinh\left(\frac{x}{a}\right)$

We substitute into the equation

$$s = \int_{-b}^{b} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

$$= \int_{-b}^{b} \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)}\,\mathrm{d}x$$

$$= \int_{-b}^{b} \cosh\left(\frac{x}{a}\right)\mathrm{d}x$$

$$= \left[a\sinh\left(\frac{x}{a}\right)\right]_{-b}^{b}$$

$$= 2a\sinh\left(\frac{b}{a}\right)$$

b i If the poles are 50 metres apart, $2b = 50$ so $b = 25$. If the lowest point of the wire is 20 metres above the ground then $a - 40 = 20$ so $a = 60$ (note that we have used the fact that the minimal point of $\cosh\left(\frac{x}{a}\right)$ occurs at $x = 0$ and has a value of 1).

The length of the wire is found by substituting our a and b values into the equation found in the previous part

$$s = 2 \times 60\sinh\left(\frac{25}{60}\right)$$

$$= 120\sinh\left(\frac{5}{12}\right)$$

$$= 51.46 \text{ (4 s.f.)}$$

19 b ii We find the height at which the wire should be attached by substituting a and $x = 25$ into the equation for y in order to get

$$y = 60\cosh\left(\frac{25}{60}\right) - 40$$
$$= 25.28 \text{ (4 s.f.)}$$

20 a i In order to satisfy the condition of the wire crossing itself at $x = 0$ we must satisfy $x = 3t^3 - t = 0$. Since we don't want to include the origin for the intersection (clearly shown in diagram that the origin is a separate point on the line $x = 0$) then we only need to satisfy $3t^2 - 1 = 0$ and this is solved to give $t = \frac{1}{\sqrt{3}}$. This is the point at which the intersection occurs and so we require $k \geqslant \frac{1}{\sqrt{3}}$

ii The minimum length of wire needed to make the brooch is when $k = \frac{1}{\sqrt{3}}$. We use the parametric version of the arc length equation which uses the parametric derivatives $\frac{dx}{dt} = 9t^2 - 1$ and $\frac{dy}{dt} = 6t$

Substituting these values in gives

$$s = \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \sqrt{\left(9t^2 - 1\right)^2 + \left(6t\right)^2}\, dt$$
$$= \int_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}} \left(9t^2 + 1\right) dt$$
$$= \left[3t^3 + t\right]_{-\frac{1}{\sqrt{3}}}^{\frac{1}{\sqrt{3}}}$$
$$= \frac{4}{\sqrt{3}}$$

20 b We use the parametric version of the arc length equation which uses the parametric derivatives
$\frac{dx}{dt} = 9t^2 - 1$ and $\frac{dy}{dt} = 6t$
Substituting these values in gives

$$s = \int_{-\frac{3}{4}}^{\frac{3}{4}} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$= \int_{-\frac{3}{4}}^{\frac{3}{4}} \sqrt{\left(9t^2 - 1\right)^2 + \left(6t\right)^2}\, dt$$
$$= \int_{-\frac{3}{4}}^{\frac{3}{4}} \left(9t^2 + 1\right) dt$$
$$= \left[3t^3 + t\right]_{-\frac{3}{4}}^{\frac{3}{4}}$$
$$= \frac{129}{32}$$
$$= 4.03 \text{ cm (3 s.f.)}$$

21 In order to find the area of the roof, we must find the arc length and multiply by the length of the hangar (300 feet). In order to find the arc length we first compute $\frac{dy}{dx} = -\sinh(0.05x)$ and then substitute into the arc length equation

$$s = \int_{-25}^{25} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_{-25}^{25} \sqrt{1 + \sinh^2\left(0.05x\right)}\, dx$$
$$= \int_{-25}^{25} \cosh\left(0.05x\right) dx$$
$$= \left[20\sinh\left(0.05x\right)\right]_{-25}^{25}$$
$$= 40\sinh\left(1.25\right)$$

64.08 feet (4 sf).

So the area is $A = 64.08 \times 300 = 19\,224$ square feet and so finally the cost of the foam is approximately $19\,224 \times 17 \approx £327\,000$.

Challenge

We compute the same as always. Note that we do not need to explicitly need to calculate the integral in $\text{f}(x)$ since we are going to differentiate anyway. We will write

$$\text{f}(x)=\int_1^x \sqrt{t^3-1}\,\text{d}t=\left[\text{F}(t)\right]_1^x=\text{F}(x)-\text{F}(1)$$

where $\text{F}(1)$ is just going to be a constant and $\text{F}(x)=\int \sqrt{x^3-1}\,\text{d}x$.

Now we find the expression $\frac{\text{d}y}{\text{d}x}=\text{f}'(x)=\text{F}'(x)=\sqrt{x^3-1}$ in order to substitute into the equation for arc length and find that

$$\begin{aligned}
s&=\int_1^4 \sqrt{1+\left(\frac{\text{d}y}{\text{d}x}\right)^2}\,\text{d}x\\
&=\int_1^4 \sqrt{1+\left(\sqrt{x^3-1}\right)^2}\,\text{d}x\\
&=\int_1^4 x^{\frac{3}{2}}\,\text{d}x\\
&=\left[\frac{x^{\frac{5}{2}}}{2.5}\right]_1^4\\
&=\frac{2^5-1}{2.5}\\
&=\frac{62}{5}\\
&=12.4
\end{aligned}$$