Poisson distributions Mixed exercise 2

1 a Let X be the number of accidents in a one-month period. Assume a Poisson distribution. So $X \sim \text{Po}(0.7)$

$P(X=0) = e^{-0.7} = 0.4966$ (4 d.p.)

b Let Y be the number of accidents in a three-month period. So $Y \sim \text{Po}(2.1)$

$$P(Y=2) = \frac{e^{-2.1} \times 2.1^2}{2!} = 0.2700 \text{ (4 d.p.)}$$

c Let A be the number of accident-free months in a sample of six months. So $A \sim B(6, 0.4966)$ from part **a**

$$P(A=2) = \binom{6}{2} \times (0.4966)^2 \times (0.5034)^4 = 0.2376 \text{ (4 d.p.)}$$

2 a X must have a constant mean rate so that the mean number of misprints in a sample is proportional to the number of chapters. Misprints must occur independently of one another and be distinct (i.e. can be counted singly in the text).

b The model is $X \sim \text{Po}(2.25)$. Find $P(X \leqslant 1)$ using a calculator:

$P(X \leqslant 1) = 0.3425$

c Let Y be the number of misprints in two randomly chosen chapters. So $Y \sim \text{Po}(4.5)$

As $\lambda = 4.5$, the required value can be found from the tables in the textbook:

$$P(Y > 6) = 1 - P(Y \leqslant 6)$$
$$= 1 - 0.8311 = 0.1689$$

3 As $Y \sim \text{Po}(\lambda)$ then $P(Y=5) = \frac{e^{-\lambda}\lambda^5}{5!}$ and $P(Y=3) = \frac{e^{-\lambda}\lambda^3}{3!}$

So as $P(Y=5) = 1.25 \times P(Y=3)$

$$\frac{e^{-\lambda}\lambda^5}{120} = 1.25 \times \frac{e^{-\lambda}\lambda^3}{6}$$
$$\lambda^2 = 25$$
$$\lambda = 5 \qquad \text{(since } \lambda \text{ must be positive)}$$

4 a The event (receipt of an email) has a constant mean rate through time. Emails are received singly and independently of each other.

b i Let X be the number of emails received in a 10-minute period. Assume a Poisson distribution, so $X \sim \text{Po}(6)$

$$P(X=7) = \frac{e^{-6} \times 6^7}{7!} = 0.1377 \text{ (4 d.p.)}$$

ii Using the tables:

$P(X \geqslant 8) = 1 - P(X \leqslant 7) = 1 - 0.7440 = 0.2560$

5 a A binomial distribution $B(n, p)$ may be approximated by a Poisson distribution $\text{Po}(np)$ when n is large and p is small, and typically $np \leqslant 10$.

b $X \sim B(50, 0.08)$

$$P(X \leqslant 3) = \binom{50}{3}(0.08)^3(0.92)^{47} + \binom{50}{2}(0.08)^2(0.92)^{48} + \binom{50}{1}(0.08)(0.92)^{49} + \binom{50}{0}(0.92)^{50}$$
$$= 0.4253 \text{ (4 d.p.)}$$

5 **c** Use as an approximation $X \sim \text{Po}(50 \times 0.08)$, i.e. $X \sim \text{Po}(4)$

Using the tables:

$\text{P}(X \leqslant 3) = 0.4335$

d Percentage error $= \dfrac{0.4335 - 0.4253}{0.4253} \times 100\% = 1.93\%$ (2 d.p)

6 $\text{P}(Y > 10) < 0.1$ so $\text{P}(Y \leqslant 10) > 0.9$

From the tables, for $\lambda = 7$, $\text{P}(Y \leqslant 10) = 0.9015$

For $\lambda = 8$, $\text{P}(Y \leqslant 10) = 0.8159$

So the largest integer value for λ satisfying the given condition is $\lambda = 7$

7 **a** Let X be the number of cuttings taking root in a sample of 20. So $X \sim \text{B}(20, 0.075)$

i $\text{P}(X = 2) = \binom{20}{2}(0.075)^2(0.925)^{18} = 0.2627$ (4 d.p.)

ii $\text{P}(X > 4) = 1 - \text{P}(X \leqslant 4)$

$= 1 - 0.9858 = 0.0142$

b Let Y be the number of cuttings taking root in a sample of 80. Then $Y \sim \text{B}(80, 0.075)$ and this can be approximated by $Y \sim \text{Po}(80 \times 0.075)$, i.e. $Y \sim \text{Po}(6)$

Using the tables:

$\text{P}(Y \geqslant 8) = 1 - \text{P}(Y \leqslant 7)$

$= 1 - 0.7440 = 0.2560$

8 Let X be the number of fish caught in a two-hour period. So $X \sim \text{Po}(4)$. Use this to find the probability that the angler catches at least 5 fish on a two-hour fishing trip.

$\text{P}(X \geqslant 5) = 1 - \text{P}(X \leqslant 4) = 1 - 0.6288 = 0.3712$

Then let Y be the number of trips in which the angle catches at least 5 fish from a sample of 5 trips. So using the probability for catching at least 5 fish on a single trip, $Y \sim \text{B}(5, 0.3712)$

$\text{P}(Y = 3) = \binom{5}{3}(0.3712)^3(0.6288)^2 = 0.2022$ (4 d.p.)

9 **a** Let X be the number of cherries in a cake. So $X \sim \text{Po}(2.5)$

i $\text{P}(X = 4) = \dfrac{e^{-2.5} \times 2.5^4}{4!} = 0.1336$ (4 d.p.)

ii Using the tables:

$\text{P}(X \geqslant 3) = 1 - \text{P}(X \leqslant 2)$

$= 1 - 0.5438 = 0.4562$

b Let Y be the number of cherries in 4 cakes. So $Y \sim \text{Po}(10)$

Using the tables:

$\text{P}(Y > 12) = 1 - \text{P}(X \leqslant 12)$

$= 1 - 0.7916 = 0.2084$

c Let A be the number of packets containing more than 12 cherries in a sample of 8 packets. So, using the result from **b**, $A \sim \text{B}(8, 0.2084)$

$\text{P}(A = 2) = \binom{8}{2}(0.2084)^2(0.7916)^6 = 0.2992$ (4 d.p.)

10 a Let X be the number of cars sold in a week. A plausible model for number of cars sold in a week would be $X \sim \text{Po}(6)$.
It may be supposed that sales are independent of each other and occur singly (assuming the salesman does not supply businesses); the constant mean rate is consistent with a Poisson model.

b $\text{P}(X = 5) = \dfrac{e^{-6} \times 6^5}{5!} = 0.1606 \text{ (4 d.p.)}$

c Let Y be the number of weeks in which the salesman sells exactly 5 cars, in a sample of 4 consecutive weeks. So, using the result from **b**, $Y \sim \text{B}(4, 0.1606)$

$$\text{P}(Y = 2) = \binom{4}{2}(0.1606)^2(0.8394)^2 = 0.1090 \text{ (4 d.p.)}$$

11 Assuming a Poisson distribution for each, A with and B being the number of letters received by Abbie and Ben in a given day (respectively): $A \sim \text{Po}(1.2)$ and $B \sim \text{Po}(0.8)$ so $A + B \sim \text{Po}(2)$

a $\text{P}(A \geqslant 1 \text{ and } B \geqslant 1) = \text{P}(A \geqslant 1) \times \text{P}(B \geqslant 1)$ (independence)

$$= (1 - \text{P}(A = 0)) \times (1 - \text{P}(B = 0))$$
$$= (1 - e^{-1.2}) \times (1 - e^{-0.8})$$
$$= (1 - 0.30119) \times (1 - 0.44933)$$
$$= 0.69881 \times 0.55067 = 0.3848 \text{ (4 d.p.)}$$

b $\text{P}(A + B = 3) = \dfrac{e^{-2} \times 2^3}{3!} = 0.1804 \text{ (4 d.p.)}$

c Let Y be the number of days on which they receive a total of 3 letters between them, from a sample of 5 days. So, using the result from **b**, $Y \sim \text{B}(5, 0.1804)$

$$\text{P}(Y \geqslant 3) = 1 - \text{P}(Y \leqslant 2)$$
$$= 1 - 0.9560 = 0.0440$$

12 Let X be the number of desktops sold in a day and Y the number of laptops sold in a day. Assuming Poisson distributions for both and that sales of desktops and laptops are independent: $X \sim \text{Po}(2.4)$ and $Y \sim \text{Po}(1.6)$ so $X + Y \sim \text{Po}(4)$

a $\text{P}(X \geqslant 2 \text{ and } Y \geqslant 2) = \text{P}(X \geqslant 2) \times \text{P}(Y \geqslant 2)$ (independence)

$$= (1 - \text{P}(X \leqslant 1)) \times (1 - \text{P}(Y \leqslant 1))$$
$$= (1 - 0.30844) \times (1 - 0.52493)$$
$$= 0.69156 \times 0.47507 = 0.3285 \text{ (4 d.p.)}$$

b $\text{P}(X + Y = 6) = \dfrac{e^{-4} \times 4^6}{6!} = 0.1042 \text{ (4 d.p.)}$

c Let A be the combined total of computer sales in a two-day period. So $A \sim \text{Po}(8)$ and the required value can be found from the tables:

$\text{P}(A \leqslant 6) = 0.3134$

13 a Let X be the number of booked passengers not turning up for the flight. Assuming independence between booked passengers, $X \sim \text{B}(150, 0.04)$
(Note: independence of events is doubtful in this example; consider reasons why this may be the case.)

b Approximate the binomial by $X \sim \text{Po}(150 \times 0.04)$, i.e. $X \sim \text{Po}(6)$ and use tables to find:

$\text{P}(X \leqslant 1) = 0.0174$

13 c $P(X \geqslant 3) = 1 - P(X \leqslant 2)$

$= 1 - 0.0620 = 0.9380$

14 a Let X be the number of misdirected calls in a sample of 10 consecutive calls. Assuming independence between calls, $X \sim B(10, 0.02)$

$P(X > 1) = 1 - P(X \leqslant 1)$

$= 1 - \binom{10}{1}(0.02)^1(0.98)^9 - \binom{10}{0}(0.98)^{10}$

$= 0.0162$ (4 d.p.)

b Let Y be the number of misdirected calls in a sample of 500 calls. Again assuming independence between calls, $X \sim B(10, 0.02)$

$E(Y) = 500 \times 0.02 = 10$ and $Var(Y) = 500 \times 0.02 \times 0.98 = 9.8$

c Approximating the binomial with a Poisson distribution, $Y \sim Po(10)$, and finding the required value from tables:

$P(Y \leqslant 7) = 0.2202$

15 a Let X be the number of people with the disease in a random sample of 10 people. So $X \sim B(10, 0.025)$

$P(X = 2) = \binom{10}{2}(0.025)^2(0.975)^8 = 0.0230$ (4 d.p.)

b Let Y be the number of people with the disease in a random sample of 120 people. So $Y \sim B(120, 0.025)$

$E(Y) = 120 \times 0.025 = 3$ and $Var(Y) = 120 \times 0.025 \times 0.975 = 2.925$

c Approximating the binomial with a Poisson distribution, $Y \sim Po(3)$, and finding the required value from tables:

$P(Y > 6) = 1 - P(Y \leqslant 6)$

$= 1 - 0.9665 = 0.0335$

16 a Assuming that accidents can be modelled using a Poisson distribution (so that the mean number of accidents in a given period of time will be proportional to the length of time), let X be the number of accidents in a six-month period. So $X \sim Po(7.5)$, and from the tables:

$P(X \leqslant 4) = 0.1321$

b Let Y be the number of accidents in a single month. So $Y \sim Po(1.25)$

$P(Y \geqslant 1) = 1 - P(Y = 0)$

$= 1 - e^{-1.25} = 1 - 0.2865 = 0.7135$ (4 d.p.)

c Let A be the number of months in which there is at least one accident, out of a sample of 6 months. From part b, $A \sim B(6, 0.7135)$

$P(A = 4) = \binom{6}{4}(0.7135)^4(0.2865)^2 = 0.3191$ (4 d.p.)

17 a Assume a Poisson distribution for breakdowns. Let X be the number of breakdowns in a single month, so $X \sim \text{Po}\left(\frac{2}{3}\right)$

$$\text{P}(X=2)=\frac{e^{-\frac{2}{3}}\times\left(\frac{2}{3}\right)^2}{2!}=0.1141 \text{ (4 d.p.)}$$

b Let Y be the number of months in which there are at least 2 breakdowns.

$\text{P}(X \geqslant 2)=1-\text{P}(X \leqslant 1)=1-0.8557=0.1443$

So $Y \sim \text{B}(4, 0.1443)$

$$\text{P}(Y=3)=\binom{4}{3}(0.1443)^3(0.8557)^1=0.0103 \text{ (4 d.p.)}$$

18 a Visits can be counted singly; assuming visits are independent and at a constant average rate, they may be modelled by a Poisson distribution; the rate of 240 per hour would then scale to a mean rate of 4 in any given minute. Let X be the number of visits in a single minute. So $X \sim \text{Po}(4)$

b $\text{P}(X=8)=\dfrac{e^{-4}\times 4^8}{8!}=0.0298 \text{ (4 d.p.)}$

c Let Y be the number of visits in a two-minute period. So $Y \sim \text{Po}(8)$

Using the tables:

$$\begin{aligned}\text{P}(Y \geqslant 10)&=1-\text{P}(Y \leqslant 9)\\&=1-0.7166=0.2843\end{aligned}$$

19 a Let X be the number of policies sold in the week.

$$\bar{x}=\frac{\sum fx}{\sum f}=\frac{10\times0+23\times1+35\times2+33\times3+24\times4+14\times5+7\times6+3\times7+1\times8}{10+23+35+33+24+14+7+3+1}=2.86$$

$$\sigma^2=\frac{\sum fx^2}{\sum f}-\left(\bar{x}\right)^2$$

$$=\frac{10\times0^2+23\times1^2+35\times2^2+33\times3^2+24\times4^2+14\times5^2+7\times6^2+3\times7^2+1\times8^2}{10+23+35+33+24+14+7+3+1}-2.86^2$$

$$=2.867 \text{ (3 d.p.)}$$

b The values of mean and variance are the same, to 2 significant figures, which would support the validity of a Poisson model.

c Model $X \sim \text{Po}(2.9)$ and by calculator:

$\text{P}(X \leqslant 2)=0.4460 \text{ (4 d.p.)}$

Challenge

a Assuming the number of planes landing in a given period of time can be modelled by a Poisson distribution, let X be the number of planes landing between 2 pm and 2:30 pm and let Y be the number of planes landing between 2.30 pm and 3 pm. Then $X \sim \text{Po}(7.5)$, $Y \sim \text{Po}(7.5)$ and $X + Y \sim \text{Po}(15)$

The solution uses the formula for conditional probability: $\text{P}(A \mid B) = \dfrac{\text{P}(A \text{ and } B)}{\text{P(B)}}$

$$\text{P}(X = 5 \mid X + Y = 10) = \frac{\text{P}(X = 5 \text{ and } Y = 5)}{\text{P}(X + Y = 10)}$$

$$= \frac{\dfrac{e^{-7.5} \times 7.5^5}{5!} \times \dfrac{e^{-7.5} \times 7.5^5}{5!}}{\dfrac{e^{-15} \times 15^{10}}{10!}}$$

$$= \frac{10!}{(5!)^2} \times \frac{7.5^{10}}{15^{10}} = \frac{10 \times 9 \times 8 \times 7 \times 6}{5 \times 4 \times 3 \times 2} \times \left(\frac{1}{2}\right)^{10} = \frac{3 \times 2 \times 7 \times 6}{2^{10}} = \frac{252}{2^{10}} = \frac{63}{2^8}$$

$$= \frac{63}{256} = 0.2461 \text{ (4 d.p.)}$$

b $$\text{P}(X > 7 \mid X + Y = 10) = \frac{\text{P}(X > 7 \text{ and } X + Y = 10)}{\text{P}(X + Y = 10)}$$

$$= \frac{\text{P}(X = 8 \text{ and } Y = 2) + \text{P}(X = 9 \text{ and } Y = 1) + \text{P}(X = 10 \text{ and } Y = 0)}{\text{P}(X + Y = 10)}$$

$$= \frac{\dfrac{e^{-7.5} \times 7.5^8}{8!} \times \dfrac{e^{-7.5} \times 7.5^2}{2!}}{\dfrac{e^{-15} \times 15^{10}}{10!}} + \frac{\dfrac{e^{-7.5} \times 7.5^9}{9!} \times \dfrac{e^{-7.5} \times 7.5^1}{9!}}{\dfrac{e^{-15} \times 15^{10}}{10!}} + \frac{\dfrac{e^{-7.5} \times 7.5^{10}}{10!} \times \dfrac{e^{-7.5} \times 7.5^0}{0!}}{\dfrac{e^{-15} \times 15^{10}}{10!}}$$

$$= \frac{7.5^{10}}{15^{10}} \times 10! \times \left(\frac{1}{2 \times 8!} + \frac{1}{9!} + \frac{1}{10!}\right) = \frac{1}{2^{10}} \times (45 + 10 + 1)$$

$$= \frac{7}{128} = 0.0547 \text{ (4 d.p.)}$$

An alternative approach to this problem is to treat the 10 landings as each independently having a 0.5 chance of being in the first or second half of the hour and modelling them as a binomial. Let A be the number of landings in the first half hour (2 pm to 2:30 pm) and $A \sim \text{B}(10, 0.5)$. To answer part **a**, find $\text{P}(A = 5)$; to answer part **b**, find $\text{P}(A > 7)$.

$$\text{P}(A = 5) = \binom{10}{5}(0.5)^5(0.5)^5 = \frac{10!}{5! \times 5!} \times \frac{1}{2^{10}} = \frac{63}{256}$$

$$\text{P}(A > 7) = \binom{10}{8}(0.5)^8(0.5)^2 + \binom{10}{9}(0.5)^9(0.5)^1 + \binom{10}{10}(0.5)^{10} = \frac{1}{2^{10}} \times 10! \times \left(\frac{1}{2 \times 8!} + \frac{1}{9!} + \frac{1}{10!}\right) = \frac{7}{128}$$