Methods in calculus – mixed exercise 3

1 a $u = e^x$, so $du = e^x \mathrm{d}x$

$$\int \frac{1}{e^x + e^{-x}} \mathrm{d}x = \int \frac{1}{e^x + \frac{1}{e^x}} \mathrm{d}x$$

$$= \int \frac{e^x}{e^{2x} + 1} \mathrm{d}x$$

$$= \int \frac{1}{u^2 + 1} du$$

$$= \arctan u + c$$

$$= \arctan e^x + c$$

b $\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} \mathrm{d}x$

$$= \int_{-\infty}^{0} \frac{1}{e^x + e^{-x}} \mathrm{d}x + \int_{0}^{\infty} \frac{1}{e^x + e^{-x}} \mathrm{d}x$$

Consider $\int_{-\infty}^{0} \frac{1}{e^x + e^{-x}} \mathrm{d}x$

$$= \lim_{t \to -\infty} \int_{t}^{0} \frac{1}{e^x + e^{-x}} \mathrm{d}x$$

$$= \lim_{t \to -\infty} \left[\arctan e^x \right]_t^0$$

$$= \lim_{t \to -\infty} \left(\frac{\pi}{4} - \arctan e^t \right)$$

$\arctan e^t \to 0$ as $t \to -\infty$, so

$$\int_{-\infty}^{0} \frac{1}{e^x + e^{-x}} dx = \frac{\pi}{4}$$

Similarly, consider $\int_{0}^{\infty} \frac{1}{e^x + e^{-x}} \mathrm{d}x$

$$= \lim_{t \to \infty} \int_{0}^{t} \frac{1}{e^x + e^{-x}} \mathrm{d}x$$

$$= \lim_{t \to \infty} \left[\arctan e^x \right]_0^t$$

$$= \lim_{t \to \infty} \left(\arctan e^t - \frac{\pi}{4} \right)$$

$\arctan e^t \to \frac{\pi}{2}$ as $t \to \infty$, so

$$\int_{0}^{\infty} \frac{1}{e^x + e^{-x}} \mathrm{d}x = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

1 b So, $\int_{-\infty}^{\infty} \frac{1}{e^x + e^{-x}} \mathrm{d}x$

$$= \int_{-\infty}^{0} \frac{1}{e^x + e^{-x}} \mathrm{d}x + \int_{0}^{\infty} \frac{1}{e^x + e^{-x}} \mathrm{d}x$$

$$= \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

2 $\mathrm{f}(x) = \frac{1 - \cos x}{\sin^2 x} = \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x}$

$$\int \mathrm{f}(x) \mathrm{d}x = \int \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \mathrm{d}x$$

Consider $\int \frac{1}{\sin^2 x} \mathrm{d}x$

$$= \int \csc^2 x dx$$

$$= -\cot x + c_1$$

Similarly, consider $\int \frac{\cos x}{\sin^2 x} \mathrm{d}x$

Let $u = \sin x$, so $du = \cos x \mathrm{d}x$

$$\int \frac{\cos x}{\sin^2 x} \mathrm{d}x = \int \frac{1}{u^2} du$$

$$= -\frac{1}{u} + c_2$$

$$= -\frac{1}{\sin x} + c_2$$

$$= -\csc x + c_2$$

Therefore,

$$\int \mathrm{f}(x) \mathrm{d}x = \int \frac{1}{\sin^2 x} - \frac{\cos x}{\sin^2 x} \mathrm{d}x$$

$$= -\cot x + \csc x + c$$

The mean value of $\mathrm{f}(x)$ over the interval $\left[\frac{\pi}{6}, \frac{\pi}{3} \right]$ is:

$$\frac{1}{\frac{\pi}{3} - \frac{\pi}{6}} \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \mathrm{f}(x) \mathrm{d}x$$

$$= \frac{6}{\pi} \left[-\cot x + \csc x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$= \frac{6}{\pi} \left(-\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} + \sqrt{3} - 2 \right)$$

$$= \frac{6}{\pi} \left(\frac{4}{\sqrt{3}} - 2 \right)$$

3 $\mathrm{f}(x) = x\sin 2x$

Consider $\int x\sin 2x\mathrm{d}x$

Integrating by parts,

$$= -\frac{x\cos 2x}{2} + \int \frac{\cos 2x}{2}\mathrm{d}x$$

$$= -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4} + c$$

The mean value of $\mathrm{f}(x)$ over the interval $\left[0, \frac{\pi}{2}\right]$ is:

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \mathrm{f}(x)\mathrm{d}x$$

$$= \frac{2}{\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

4 a $y = \arccos x^2$

Let $t = x^2$, so $\frac{\mathrm{d}t}{\mathrm{d}x} = 2x$

So, $y = \arccos t$

$\cos y = t$

$$-\sin y\frac{\mathrm{d}y}{\mathrm{d}t} = 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{\sin y}$$

$$= -\frac{1}{\sqrt{1-\cos^2 y}}$$

$$= -\frac{1}{\sqrt{1-x^4}}$$

Using chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{2x}{\sqrt{1-x^4}}$$

b $\int \frac{3x}{\sqrt{16-x^4}}dx$

$$= \frac{1}{4}\int \frac{3x}{\sqrt{1-\left(\frac{x}{2}\right)^4}}\mathrm{d}x$$

4 b (Let $u = \frac{x}{2}$, so $du = \frac{1}{2}\mathrm{d}x$)

$$= \frac{3}{2}\int \frac{2u}{\sqrt{1-u^4}}du$$

$$= -\frac{3}{2}\arccos u^2 + c$$

$$= -\frac{3}{2}\arccos\left(\frac{x^2}{4}\right) + c$$

5 a $y = \mathrm{f}(x) = \arctan\left(\frac{2x+3}{x-1}\right)$

Let $t = \frac{2x+3}{x-1}$

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{(x-1)2-(2x+3)}{(x-1)^2}$$

$$= -\frac{5}{(x-1)^2}$$

Also, $y = \arctan t$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{1+t^2}$$

$$= \frac{1}{1+\left(\frac{2x+3}{x-1}\right)^2}$$

$$= \frac{(x-1)^2}{(x-1)^2+(2x+3)^2}$$

$$= \frac{(x-1)^2}{5(x^2+2x+2)}$$

Using chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}$$

$$= \frac{(x-1)^2}{5(x^2+2x+2)}\times -\frac{5}{(x-1)^2}$$

$$= -\frac{1}{x^2+2x+2}$$

b $x^2+2x+2 = (x+1)^2+1$

So $(x+1)^2+1 \geq 1$

Hence,

$$|f'(x)| = \left|-\frac{1}{x^2+2x+2}\right| \leqslant 1$$

6 a We say an integral $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ is improper if one or both of the limits are infinite or if $\mathrm{f}(x)$ is undefined at some point in the domain $[a,b]$.

b $\int_0^\infty \frac{1}{(x+1)\sqrt{x}}\,\mathrm{d}x$

The given interval is improper because the upper limit is infinite and also because the integrand is undefined at the lower limit $x=0$.

c $y=\arctan\sqrt{x}$

Let $t=\sqrt{x}$, so $\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}$

So, $y=\arctan t$

$\tan y = t$

$\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}t}=1$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x}$$

Using chain rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}=\frac{1}{2(x+1)\sqrt{x}}$$

$$\int_0^\infty \frac{1}{(x+1)\sqrt{x}}=\left(\int_0^1 \frac{1}{(x+1)\sqrt{x}}\,\mathrm{d}x+\int_1^\infty \frac{1}{(x+1)\sqrt{x}}\,\mathrm{d}x\right)$$

Consider $\int_0^1 \frac{1}{(x+1)\sqrt{x}}\,\mathrm{d}x$

$$=\lim_{t\to 0}\int_t^1 \frac{1}{(x+1)\sqrt{x}}\,\mathrm{d}x$$

$$=\lim_{t\to 0}\left[2\arctan\sqrt{x}\right]_t^1$$

$$=\lim_{t\to 0}\left(\frac{\pi}{2}-2\arctan\sqrt{t}\right)=\frac{\pi}{2}$$

6 c $\arctan\sqrt{t}\to 0$ as $t\to 0$, so the integral converges.

Consider $\int_1^\infty \frac{1}{(x+1)\sqrt{x}}\,\mathrm{d}x$

$$=\lim_{t\to\infty}\int_1^t \frac{1}{(x+1)\sqrt{x}}\,\mathrm{d}x$$

$$=\lim_{t\to\infty}\left[2\arctan\sqrt{x}\right]_1^t$$

$$=\lim_{t\to 0}\left(2\arctan\sqrt{t}-\frac{\pi}{2}\right)=\frac{\pi}{2}$$

$\arctan\sqrt{t}\to\frac{\pi}{2}$ as $t\to\infty$, so the integral converges.

Since both integrals converge,

$\int_0^\infty \frac{1}{(x+1)\sqrt{x}}$ converges, and

$$\int_0^\infty \frac{1}{(x+1)\sqrt{x}}=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

7 $\mathrm{f}(x)=\frac{1+5x}{\sqrt{1-5x^2}}$

$$\int \mathrm{f}(x)\,\mathrm{d}x=\int\frac{1+5x}{\sqrt{1-5x^2}}\,\mathrm{d}x=\int\frac{1}{\sqrt{1-5x^2}}\,\mathrm{d}x+\int\frac{5x}{\sqrt{1-5x^2}}\,\mathrm{d}x$$

Consider $\int\frac{1}{\sqrt{1-5x^2}}\,\mathrm{d}x$

$$=\frac{1}{\sqrt{5}}\int\frac{1}{\sqrt{\left(\frac{1}{\sqrt{5}}\right)^2-x^2}}\,\mathrm{d}x$$

$$=\frac{1}{\sqrt{5}}\arcsin\left(\frac{x}{\frac{1}{\sqrt{5}}}\right)+c_1$$

$$=\frac{1}{\sqrt{5}}\arcsin\left(\sqrt{5}x\right)+c_1$$

Consider $\int\frac{5x}{\sqrt{1-5x^2}}\,\mathrm{d}x$

Let $u=1-5x^2$ and $du=-10x\mathrm{d}x$

7 $\int \frac{5x}{\sqrt{1-5x^2}}\mathrm{d}x = -\frac{1}{2}\int \frac{1}{\sqrt{u}}du$

$= -\sqrt{u} + c_2$

$= -\sqrt{1-5x^2} + c_2$

Therefore,

$\int \mathrm{f}(x)\mathrm{d}x$

$= \int \frac{1}{\sqrt{1-5x^2}}\mathrm{d}x + \int \frac{5x}{\sqrt{1-5x^2}}\mathrm{d}x$

$= \frac{1}{\sqrt{5}}\arcsin\left(\sqrt{5}x\right) - \sqrt{1-5x^2} + c$

Therefore, $A = -1$ and $B = \frac{1}{\sqrt{5}}$

8 a Consider $\int \frac{1}{x^2+1}\mathrm{d}x$

Let $x = \tan\theta$, $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$

$1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$

So,

$\int \frac{1}{x^2+1}\mathrm{d}x$

$= \int \frac{\sec^2\theta}{\sec^2\theta}d\theta$

$= \theta + c$

$= \arctan x + c$

Therefore,

$\int_0^t \frac{1}{1+x^2}\mathrm{d}x$

$= \left[\arctan x\right]_0^t$

$= \arctan t$

b i $\int_0^\infty \frac{1}{1+x^2}\mathrm{d}x$

$= \lim_{t\to\infty}\int_0^t \frac{1}{1+x^2}\mathrm{d}x$

$= \lim_{t\to\infty}\left[\arctan x\right]_0^t$

$= \lim_{t\to\infty}\arctan t$

$\arctan t \to \frac{\pi}{2}$ as $t \to \infty$, so

$\int_0^\infty \frac{1}{1+x^2}\mathrm{d}x = \frac{\pi}{2}$

8 b ii $\int_{-\infty}^\infty \frac{1}{1+x^2}\mathrm{d}x$

$= \int_{-\infty}^0 \frac{1}{1+x^2}\mathrm{d}x + \int_0^\infty \frac{1}{1+x^2}\mathrm{d}x$

Consider $\int_{-\infty}^0 \frac{1}{1+x^2}\mathrm{d}x$

$= \lim_{t\to-\infty}\int_t^0 \frac{1}{1+x^2}\mathrm{d}x$

$= \lim_{t\to-\infty}\left[\arctan x\right]_t^0$

$= \lim_{t\to-\infty}\left(-\arctan t\right)$

$\arctan t \to -\frac{\pi}{2}$ as $t \to -\infty$, so

$\int_{-\infty}^0 \frac{1}{1+x^2}\mathrm{d}x = \frac{\pi}{2}$

Therefore,

$\int_{-\infty}^\infty \frac{1}{1+x^2}\mathrm{d}x = \frac{\pi}{2} + \frac{\pi}{2} = \pi$

9 a $\mathrm{f}(x)=\dfrac{1+2x}{1+4x^2}$

$\int \mathrm{f}(x)\,\mathrm{d}x=\int \dfrac{1+2x}{1+4x^2}\,\mathrm{d}x$

$=\int \dfrac{1}{1+4x^2}\,\mathrm{d}x+\int \dfrac{2x}{1+4x^2}\,\mathrm{d}x$

Consider $\int \dfrac{1}{1+4x^2}\,\mathrm{d}x$

$=\dfrac{1}{4}\int \dfrac{1}{\left(\frac{1}{2}\right)^2+x^2}\,\mathrm{d}x$

$=\dfrac{1}{4}\left(2\arctan\left(\dfrac{x}{\frac{1}{2}}\right)\right)+c_1$

$=\dfrac{1}{2}\arctan 2x+c_1$

Similarly, consider $\int \dfrac{2x}{1+4x^2}\,\mathrm{d}x$

Let $u=1+4x^2$ and $du=8x\mathrm{d}x$

$\int \dfrac{2x}{1+4x^2}\,\mathrm{d}x=\dfrac{1}{4}\int \dfrac{1}{u}\,du$

$=\dfrac{1}{4}\ln u+c_2$

$=\dfrac{1}{4}\ln\left(1+4x^2\right)+c_2$

Therefore,

$\int \mathrm{f}(x)\,\mathrm{d}x$

$=\int \dfrac{1}{1+4x^2}\,\mathrm{d}x+\int \dfrac{2x}{1+4x^2}\,\mathrm{d}x$

$=\dfrac{1}{2}\arctan 2x+\dfrac{1}{4}\ln\left(1+4x^2\right)+c$

Therefore $A=\dfrac{1}{4}$ and $B=\dfrac{1}{2}$

b $\displaystyle\int_0^{0.5} f(x)\,\mathrm{d}x$

$=\left[\dfrac{1}{2}\arctan 2x+\dfrac{1}{4}\ln\left(1+4x^2\right)\right]_0^{0.5}$

$=\dfrac{\pi}{8}+\dfrac{1}{4}\ln 2=\dfrac{1}{8}(\pi+2\ln 2)$

10 a $\int \dfrac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x$

$=\dfrac{1}{\sqrt{9}}\int \dfrac{1}{\sqrt{\left(\sqrt{\frac{9}{4}}\right)^2-x^2}}\,\mathrm{d}x$

$=\dfrac{1}{3}\arcsin\left(\dfrac{x}{\frac{2}{3}}\right)+c$

$=\dfrac{1}{3}\arcsin\left(\dfrac{3}{2}x\right)+c$

Hence $P=\dfrac{1}{3}$ and $Q=\dfrac{3}{2}$

b $\displaystyle\int_0^{\frac{2}{3}} \dfrac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x$

$=\displaystyle\lim_{t\to\frac{2}{3}}\int_0^{t} \dfrac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x$

$=\displaystyle\lim_{t\to\frac{2}{3}}\left[\dfrac{1}{3}\arcsin\left(\dfrac{3}{2}x\right)\right]_0^{t}$

$=\displaystyle\lim_{t\to\frac{2}{3}}\dfrac{1}{3}\arcsin\left(\dfrac{3}{2}t\right)=\dfrac{\pi}{6}$

11 Consider $\int_0^{\frac{1}{2}} \frac{x^4}{\sqrt{1-x^2}}\,dx$

$x=\sin\theta$, $dx=\cos\theta\,d\theta$

$1-x^2=1-\sin^2\theta=\cos^2\theta$

So,

$$\int_0^{\frac{1}{2}} \frac{x^4}{\sqrt{1-x^2}}\,dx$$

$$=\int_0^{\frac{\pi}{6}} \sin^4\theta\,d\theta$$

Expanding $\sin^4\theta$ using de Moivre's theorem,

$$=\int_0^{\frac{\pi}{6}} \frac{\cos 4\theta-4\cos 2\theta+3}{8}\,d\theta$$

$$=\frac{1}{8}\left[\frac{1}{4}\sin 4\theta-2\sin 2\theta+3\theta\right]_0^{\frac{\pi}{6}}$$

$$=\frac{1}{8}\left(\frac{\sqrt{3}}{8}-\sqrt{3}+\frac{\pi}{2}\right)$$

$$=\frac{1}{64}\left(4\pi-7\sqrt{3}\right)$$

12 a $f(x)=\frac{x}{1+x^4}$

$$\int_0^1 f(x)\,dx=\int_0^1 \frac{x}{1+x^4}\,dx$$

$u=x^2$ and $du=2x\,dx$

So,

$$\int_0^1 \frac{x}{1+x^4}\,dx=\frac{1}{2}\int_0^1 \frac{1}{1+u^2}\,du$$

$$=\frac{1}{2}\left[\arctan u\right]_0^1=\frac{\pi}{8}$$

12 b $$\int_0^{\infty} f(x)\,dx=\lim_{t\to\infty}\int_0^t \frac{x}{1+x^4}\,dx$$

$$=\lim_{t\to\infty}\left(\frac{1}{2}\left[\arctan x^2\right]_0^t\right)$$

$$=\frac{1}{2}\lim_{t\to\infty}\arctan t^2$$

$\arctan t^2 \to \frac{\pi}{2}$ as $t\to\infty$, so the integral converges, and

$$\int_0^{\infty} f(x)\,dx=\frac{\pi}{4}$$

13 $\int \frac{2x^3-2x^2+18x+9}{x^4+9x^2}$

$$=\int \frac{2x^3-2x^2+18x+9}{x^2\left(x^2+9\right)}\,dx$$

$$=\int \frac{Ax+B}{x^2}+\frac{Cx+D}{x^2+9}\,dx$$

$2x^3-2x^2+18x+9\equiv$

$(Ax+B)\left(x^2+9\right)+(Cx+D)\left(x^2\right)$

Set $x=0$, so $B=1$

$B+D=-2\Rightarrow D=-3$

$9A=18\Rightarrow A=2$

$A+C=2\Rightarrow C=0$

Therefore,

$$\int \frac{2x^3-2x^2+18x+9}{x^4+9x^2}$$

$$=\int \frac{2}{x}\,dx+\int \frac{1}{x^2}\,dx-3\int \frac{1}{x^2+9}\,dx$$

$$=2\ln|x|-\frac{1}{x}-\arctan\left(\frac{x}{3}\right)+c$$

14 a $$f(x)=\frac{x^2-3x+14}{x^3-4x^2+2x-8}$$

$$=\frac{P}{x-4}+\frac{Q}{x^2+2}$$

$x^2-3x+14\equiv P\left(x^2+2\right)+Q(x-4)$

Set $x=4$, so $P=1$

$2P-4Q=14\Rightarrow Q=-3$

Therefore,

$$\frac{x^2-3x+14}{x^3-4x^2+2x-8}=\frac{1}{x-4}-\frac{3}{x^2+2}$$

14 b $\int f(x)\,dx$

$= \int \frac{1}{x-4}\,dx - 3\int \frac{1}{x^2+2}\,dx$

$= \ln|x-4| - \frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + c$

Therefore, $A = 1$ and $B = -\frac{3}{\sqrt{2}}$

c $\int_4^\infty f(x)\,dx = \int_4^5 f(x)\,dx + \int_5^\infty f(x)\,dx$

Consider $\int_4^5 f(x)\,dx$

$= \lim_{t\to 4}\int_t^5 \frac{1}{x-4} - \frac{3}{x^2+2}\,dx$

$= \lim_{t\to 4}\left[\ln|x-4| - \frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)\right]_t^5$

$= \lim_{t\to 4}\left(-\frac{3}{\sqrt{2}}\arctan\left(\frac{5}{\sqrt{2}}\right) - \ln|x-4| + \frac{3}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right)\right)$

$\ln|t-4| \to -\infty$ as $t \to 4$, so the integral diverges.

15 a $\int \frac{2}{x^3+x}\,dx$

$= \int \frac{2}{x(x^2+1)}\,dx$

$= \int \frac{A}{x} + \frac{Bx+C}{x^2+1}\,dx$

$2 \equiv A(x^2+1) + (Bx+C)(x)$

Set $x = 0$, so $A = 2$

$A + B = 0 \Rightarrow B = -2$

$C = 0$

Therefore,

$\int \frac{2}{x^3+x}\,dx$

$= 2\int \frac{1}{x}\,dx - 2\int \frac{x}{x^2+1}\,dx$

$= 2\ln|x| - \ln|x^2+1| + c$

$= \ln\left|\frac{x^2}{x^2+1}\right| + c$

15 b The mean value of $f(x)$ over the interval $[1,2]$ is:

$\frac{1}{2-1}\int_1^2 f(x)\,dx$

$= \left[\ln\left|\frac{x^2}{x^2+1}\right|\right]_1^2$

$= \ln\left(\frac{4}{5}\right) - \ln\left(\frac{1}{2}\right) = \ln\left(\frac{8}{5}\right).$

c The mean value of $-\frac{6}{x}$ over the interval $[1,2]$ is:

$\frac{1}{2-1}\int_1^2 -\frac{6}{x}\,dx.$

$= [-6\ln x]_1^2$

$= -6\ln 2$

$= -\ln 64.$

The mean value of $2f(x) - \frac{6}{x}$ over the interval $[1,2]$ is:

$2\ln\left(\frac{8}{5}\right) - \ln 64$

$= 2\ln 8 - 2\ln 5 - 2\ln 8$

$= -2\ln 5$

Challenge

a $\mathrm{f}(x) = x^3 - 2x + 4$

The mean value of $\mathrm{f}(x)$ over the interval $[0, 2]$ is:

$$\frac{1}{2-0}\int_1^2 x^3 - 2x + 4\,\mathrm{d}x$$

$$= \frac{1}{2}\left[\frac{x^4}{4} - x^2 + 4x\right]_0^2$$

$= 4$

The mean value $\mathrm{f}(c) = 4$

$c^3 - 2c + 4 = 4$

$c(c^2 - 2) = 0$

$c = 0, \pm\sqrt{2}$

Since c has to lie in the domain $[0, 2]$,

$c = 0$ or $c = \sqrt{2}$.

b One example is $\mathrm{f}(x) = 0$ for $x \leqslant 1$ and $\mathrm{f}(x) = 1$ for $x > 1$. This has mean value $\frac{1}{2}$ lie on the domain $[0, 2]$, but the function only attains values 0 and 1.