Exam-style practice: Paper 1

1 a First we find y^2 and $\frac{\mathrm{d}x}{\mathrm{d}t}$.

$y = t - t^2 + 1 \Rightarrow y^2 = \left(t - t^2 + 1\right)^2$

$y^2 = t^4 - 2t^3 - t^2 + 2t + 1.$

$x = t + t^2 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 1 + 2t.$

We now find the limits in terms of t.

$x = 0 \Rightarrow t = 0$

$x = 3.75 = t + t^2$

$t^2 + t - 3.75 = 0$

$t = \frac{-1 \pm \sqrt{1+15}}{2}$

We neglect the negative solution, so the upper limit is $t = \frac{3}{2}$.

$V = \pi \int_0^{1.5} \left(t^4 - 2t^3 - t^2 + 2t + 1\right)\left(1 + 2t\right)\mathrm{d}t$

$= \pi \int_0^{1.5} \left(2t^5 - 3t^4 - 4t^3 + 3t^2 + 4t + 1\right)\mathrm{d}t$

$= \pi \left[\frac{t^6}{3} - \frac{3t^5}{5} - t^4 + t^3 + 2t^2 + t\right]_0^{1.5}$

$= 11.16\,\text{unit}^3.$

Since we have that a unit is 10 cm^3, this means:

$V = 11.16 \times \left(10\,\text{cm}\right)^3$

$\approx 11\,200\,\text{cm}^3$

b A criticism of the model is that it does not take into account the thickness of the clay.

2 a Substituting $(1,4,7)$ and the coefficients of $2x + 3y - z - 10 = 0$ into

$dist = \frac{\left|ax_0 + by_0 + cz_0 + d\right|}{\sqrt{a^2 + b^2 + c^2}}$

gives

$dist = \frac{\left|(2\times1) + (3\times4) + (-1\times7) - 10\right|}{\sqrt{2^2 + 3^2 + (-1)^2}}$

$= \frac{3}{\sqrt{14}}$

$= \frac{3}{14}\sqrt{14}$

Thus $a = \frac{3}{14}$.

b Since $\mathbf{v} = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to $\mathbf{r} = \lambda\left(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}\right) + \mu\left(a\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\right)$, we know that $\mathbf{r}.\mathbf{v} = \mathbf{0}$.

$\mathbf{r}.\mathbf{v}$

$= \left(\lambda\left(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}\right) + \mu\left(a\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}\right)\right).\left(5\mathbf{i} + \mathbf{j} - \mathbf{k}\right)$

$= \left(\left(\lambda + \mu a\right)\mathbf{i} + \left(-3\lambda + 2\mu\right)\mathbf{j} + \left(2\lambda - 3\mu\right)\mathbf{k}\right).\left(5\mathbf{i} + \mathbf{j} - \mathbf{k}\right)$

$= 5\lambda + 5\mu a - 3\lambda + 2\mu - 2\lambda + 3\mu$

$= 5\mu a + 5\mu = 0$

$\Rightarrow a = -1$

c The normal vector of the plane Π_1 is $\mathbf{v}_1 = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, the plane Π_2 has a perpendicular vector $\mathbf{v}_2 = 5\mathbf{i} + \mathbf{j} - \mathbf{k}$.

$\theta = \arccos\left(\frac{\mathbf{v}_1.\mathbf{v}_2}{\left|\mathbf{v}_1\right|\left|\mathbf{v}_2\right|}\right)$

$= \arccos\left(\frac{(2\times5) + (3\times1) + (-1\times-1)}{\sqrt{2^2 + 3^2 + (-1)^2}\sqrt{5^2 + 1^2 + (-1)^2}}\right)$

$= \arccos\left(\frac{14}{\sqrt{14}\sqrt{27}}\right)$

$= \arccos\left(\frac{\sqrt{14}}{\sqrt{27}}\right)$

$= 43.9°$

3 a The base case $n=1$,

$$\begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix}=\begin{pmatrix}1&2\times1&2\times1^2\\0&1&2\times1\\0&0&1\end{pmatrix}$$ is true.

Assume the result is true for $n=k$.

For $n=k$, $$\begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix}^k=\begin{pmatrix}1&2k&2k^2\\0&1&2k\\0&0&1\end{pmatrix}$$

For $n=k+1$,

$$\begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix}^{k+1}=\begin{pmatrix}1&2&2\\0&1&2\\0&0&1\end{pmatrix}\begin{pmatrix}1&2k&2k^2\\0&1&2k\\0&0&1\end{pmatrix}$$
$$=\begin{pmatrix}1&2k+2&2k^2+4k+2\\0&1&2k+2\\0&0&1\end{pmatrix}$$
$$=\begin{pmatrix}1&2(k+1)&2(k+1)^2\\0&1&2(k+1)\\0&0&1\end{pmatrix}$$

Thus the result is true for $n=k+1$.
The result is true for $n=1$ and if it is true for $n=k$, then it is true for $n=k+1$.
By mathematical induction the result is true for all positive integers n.

b i M^{-1} only exists if the discriminant is non-zero. So for the discriminant to be zero, we find

$$discrim=1\times((-2\times1)-(0\times-1))$$
$$+k\times((0\times3)-(2\times1))$$
$$+4\times((2\times-1)-(-2\times3))$$
$$=-2-2k+16$$
$$=-2k+14=0$$
$$\Rightarrow k=7.$$

3 b ii Using

$$A^{-1}=\begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}^{-1}$$
$$=\frac{1}{|A|}\begin{pmatrix}\begin{vmatrix}e&f\\h&i\end{vmatrix}&\begin{vmatrix}c&b\\i&h\end{vmatrix}&\begin{vmatrix}b&c\\e&f\end{vmatrix}\\\begin{vmatrix}f&d\\i&g\end{vmatrix}&\begin{vmatrix}a&c\\g&i\end{vmatrix}&\begin{vmatrix}c&a\\f&d\end{vmatrix}\\\begin{vmatrix}d&e\\g&h\end{vmatrix}&\begin{vmatrix}b&a\\h&g\end{vmatrix}&\begin{vmatrix}a&b\\d&e\end{vmatrix}\end{pmatrix}$$

we may compute

$$\begin{pmatrix}1&k&4\\2&-2&0\\3&-1&1\end{pmatrix}^{-1}=\frac{1}{14-2k}\begin{pmatrix}\begin{vmatrix}-2&0\\-1&1\end{vmatrix}&\begin{vmatrix}4&k\\1&-1\end{vmatrix}&\begin{vmatrix}k&4\\-2&0\end{vmatrix}\\\begin{vmatrix}0&2\\1&3\end{vmatrix}&\begin{vmatrix}1&4\\3&1\end{vmatrix}&\begin{vmatrix}4&1\\0&2\end{vmatrix}\\\begin{vmatrix}2&-2\\3&-1\end{vmatrix}&\begin{vmatrix}k&1\\-1&3\end{vmatrix}&\begin{vmatrix}1&k\\2&-2\end{vmatrix}\end{pmatrix}$$
$$=\frac{1}{14-2k}\begin{pmatrix}-2&-(k+4)&8\\-2&-11&8\\4&3k+1&-2(k+1)\end{pmatrix}$$

4 a Using $z=\cos\theta+i\sin\theta$ (since the modulus is 1 we don't need a coefficient) we may write

$$z^n=(\cos\theta+i\sin\theta)^n$$
$$=\cos(n\theta)+i\sin(n\theta)$$

and

$$z^{-n}=(\cos\theta+i\sin\theta)^{-n}$$
$$=\cos(n\theta)-i\sin(n\theta)$$

by De Moivre's theorem.
We find the difference of these terms and get

$$z^n-z^{-n}=\cos(n\theta)+i\sin(n\theta)-(\cos(n\theta)-i\sin(n\theta))$$
$$=2i\sin(n\theta).$$

4 b We use

$\left(z-z^{-1}\right)^4=\left(2i\sin\theta\right)^4$

$=16\sin^4\theta$

(from part a with $n=1$) in order to obtain

$$8\sin^4\theta=\frac{1}{2}\left(z-z^{-1}\right)^4$$
$$=\frac{1}{2}\left(z^4-4z^2+6-4z^{-2}+z^{-4}\right)$$
$$=\frac{1}{2}\left(z^4+z^{-4}\right)-2\left(z^2+z^{-2}\right)+3$$
$$=\cos(4\theta)-4\cos(2\theta)+3.$$

Note that we have used the fact that $z^n+z^{-n}=2\cos(n\theta)$.

5 a The litres of liquid in the vat after t minutes is given by $V=500-15t+30t=500+15t$.

Let x grams be the amount of sugar in the vat after t minutes.

Concentration of sugar after t minutes is

$Conc=\frac{x}{500+15t}$ grams per litre.

The rate of the sugar into the vat is $Rate_{SugIn}=30\times25=750$ grams per minute.

The rate of the sugar out of the vat is

$Rate_{SugOut}=15\times\frac{x}{500+15t}=\frac{3x}{100+3t}$

grams per minute.

Hence we have the rate of change of sugar in grams with respect to time in minutes as $\frac{dx}{dt}=750-\frac{3x}{100+3t}$.

5 b We write down the equation above, in the form $\frac{dx}{dt}+Px=Q$.

$$\frac{dx}{dt}+\frac{3x}{100+3t}=750$$

Now use an integrating factor to solve the differential equation.

$$IF=e^{\int P dt}$$
$$=e^{\int\frac{3}{100+3t}dt}$$
$$=e^{\ln(100+3t)}$$
$$=100+3t.$$

Multiplying our original equation through by the integrating factor gives

$$(100+3t)\frac{dx}{dt}+3x=750(100+3t)$$
$$\frac{d}{dt}(100x+3tx)=75\,000+2250t.$$

Integrating both sides with respect to t gives $100x+3tx=75\,000t+1125t^2+c$.

Using the initial conditions $t=0,\ x=0$ we find that

$0+0=0+0+c$

$c=0.$

Thus the equation is

$100x+3tx=75\,000t+1125t^2$.

So after 10 minutes

$100x+30x=750\,000+112\,500$

$$x=\frac{862\,500}{130}\approx6635g$$

c The model could be refined to reflect pressure caused by the volume of oil which would effect rate of leaking. Could take into account that sugar does not disperse uniformly throughout the vat on entry.

6 a We convert $|z+12+5i|=13$ into Cartesian form and then to polar. Since we know that this locus of points is a circle with radius 13, centred at $(-12,-5)$, we know that the appropriate Cartesian equation is $(x+12)^2+(y+5)^2=169$.

We convert this equation to polar form via $x=r\cos\theta$

and

$y=r\sin\theta$,

giving

$(r\cos\theta+12)^2+(r\sin\theta+5)^2=169$

$r^2+24r\cos\theta+10r\sin\theta=0$

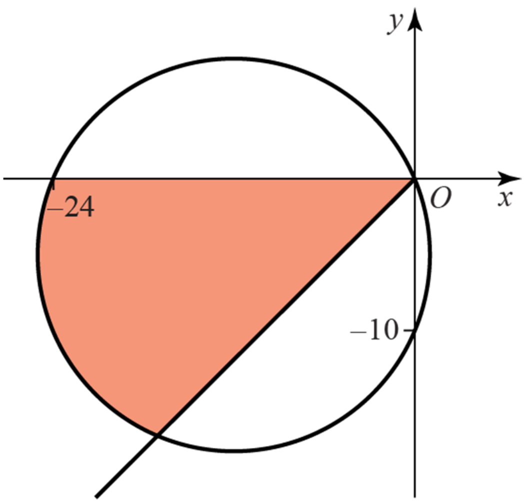
$r=-2(12\cos\theta+5\sin\theta)$ when $r\neq 0$.

b The set of points

$A=\{z:|z+12+5i|\leqslant 13\}\cap\left\{z:-\pi\leqslant \arg z\leqslant -\frac{3}{4}\pi\right\}$

defines the segment of a disc (filled circle) with radius 13, centred at $(-12,-5)$ between the half lines

$\theta=-\pi$ and $\theta=-\frac{3\pi}{4}$.



6 c To find the area of the region defined by A, we use the polar form $r=-2(12\cos\theta+5\sin\theta)$ to calculate

$$A=\frac{1}{2}\int_{-\pi}^{\frac{3\pi}{4}}(-2)^2(12\cos\theta+5\sin\theta)^2\,d\theta$$

$$=2\int_{-\pi}^{\frac{3\pi}{4}}(144\cos^2\theta+120\cos\theta\sin\theta+25\sin^2\theta)\,d\theta$$

$$=2\int_{-\pi}^{\frac{3\pi}{4}}\left(72(1+\cos 2\theta)+60\sin 2\theta+\frac{25}{2}(1-\cos 2\theta)\right)d\theta$$

$$=2\left[\left(72\left(\theta+\frac{1}{2}\sin 2\theta\right)-30\cos 2\theta+\frac{25}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)\right)\right]_{-\pi}^{\frac{3\pi}{4}}$$

$$=\left(-\frac{507\pi}{4}+\frac{119}{2}-(-169\pi-60)\right)$$

$$=\frac{169}{4}\pi+\frac{239}{2}$$

$$\approx 252\text{ units}^2$$

7 a Differentiating the equation

$$\frac{dx}{dt} = 0.3x + 0.2y + 1 \text{ with respect to } t$$

gives the expression

$$\frac{d^2x}{dt^2} = 0.3\frac{dx}{dt} + 0.2\frac{dy}{dt}.$$

We remove y from the system of equations by

$$0.3 \times \frac{dx}{dt} - 0.2 \times \frac{dy}{dt}$$
$$= 0.3 \times (0.3x + 0.2y + 1) - 0.2 \times (-0.2x + 0.3y)$$
$$= (0.3^2 + 0.2^2)x + 0.3$$

and so we can write $\frac{dy}{dt}$ in terms of x as

$$0.2\frac{dy}{dt} = 0.3\frac{dx}{dt} - ((0.3^2 + 0.2^2)x + 0.3).$$

Combining the first expression we found for $\frac{d^2x}{dt^2}$ with the expression for $\frac{dy}{dt}$, we get

$$\frac{d^2x}{dt^2} = 0.3\frac{dx}{dt} + 0.2\frac{dy}{dt}$$
$$= 0.3\frac{dx}{dt} + 0.3\frac{dx}{dt} - ((0.3^2 + 0.2^2)x + 0.3)$$
$$= 0.6\frac{dx}{dt} - (0.13x + 0.3).$$

Multiplying through by 100 and rearranging to have all terms on the left-hand side gives us

$$100\frac{d^2x}{dt^2} - 60\frac{dx}{dt} + 13x + 30 = 0$$

7 b The auxiliary equation is

$100m^2 - 60m + 13 = 0$, with solutions

$$m = \frac{60 \pm \sqrt{3600 - 5200}}{200}$$
$$= \frac{3}{10} \pm \frac{i}{5}.$$

Thus we have the complementary function

$$x_c = e^{0.3t}(A\cos 0.2t + B\sin 0.2t).$$

For the particular integral, we try $x = C$ with differentials $\frac{dx}{dt} = 0$ and $\frac{d^2x}{dt^2} = 0$.

Substituting these expressions into our differential equation gives

$$0 - 0 + 13C + 30 = 0$$
$$C = -\frac{30}{13}.$$

So the general solution for x is

$$x_G = e^{0.3t}(A\cos 0.2t + B\sin 0.2t) - \frac{30}{13}$$

c By differentiating

$$x_G = e^{0.3t}(A\cos 0.2t + B\sin 0.2t) - \frac{30}{13}$$

with respect to t, we obtain

$$\frac{dx_G}{dt} = 0.2e^{0.3t}(B\cos 0.2t - A\sin 0.2t)$$
$$+0.3e^{0.3t}(A\cos 0.2t + B\sin 0.2t)$$
$$= 0.2e^{0.3t}(B\cos 0.2t - A\sin 0.2t) + 0.3\left(x_G + \frac{30}{13}\right)$$

and since we have $\frac{dx}{dt} = 0.3x + 0.2y + 1$,

we can rewrite

$$\frac{dx_G}{dt} = 0.2e^{0.3t}(B\cos 0.2t - A\sin 0.2t) + 0.3\left(x_G + \frac{30}{13}\right)$$
$$= 0.3x_G + 0.2y_G + 1$$

After cancelling $0.3x_G$ from both sides, we solve for

$$0.2e^{0.3t}(B\cos 0.2t - A\sin 0.2t) + 0.3\left(\frac{30}{13}\right) = 0.2y_G + 1$$
$$0.2y_G = 0.2e^{0.3t}(B\cos 0.2t - A\sin 0.2t) + 0.3\left(\frac{30}{13}\right) - 1$$
$$y_G = e^{0.3t}(B\cos 0.2t - A\sin 0.2t) + \left(\frac{3}{2} \times \frac{30}{13}\right) - 5$$
$$y_G = e^{0.3t}(B\cos 0.2t - A\sin 0.2t) - \frac{20}{13}$$

7 d Using the initial conditions $t=0,\ x=10,\ y=5$ in the general solutions for x and y gives

$$10=e^{0}\left(A\cos 0+B\sin 0\right)-\frac{30}{13}$$

$$A=\frac{160}{13}$$

and

$$5=e^{0}\left(B\cos 0-A\sin 0\right)-\frac{20}{13}$$

$$B=\frac{85}{13}.$$

Thus, we have particular solutions

$$x=e^{0.3t}\left(\frac{160}{13}\cos 0.2t+\frac{85}{13}\sin 0.2t\right)-\frac{30}{13}$$

$$y=e^{0.3t}\left(\frac{85}{13}\cos 0.2t-\frac{160}{13}\sin 0.2t\right)-\frac{20}{13}$$

e $$y(3)=e^{0.9}\left(\frac{85}{13}\cos 0.6-\frac{160}{13}\sin 0.6\right)-\frac{20}{13}$$

$$\approx -5.36$$

The concentration on the right side of the model is predicted to be negative after 3 hours. This is clearly nonsense, so the model is not suitable.