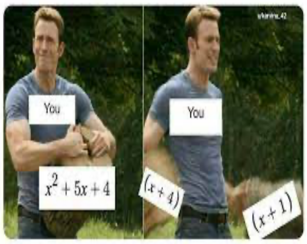
# Factorising:





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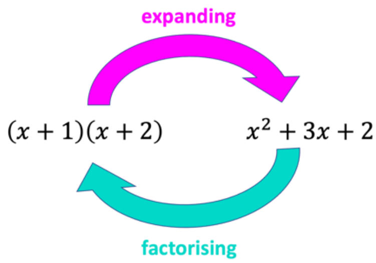
# 1 Introduction

What are factors? Numbers have factors. For example, 6= $2 \times 3$. 2 and 3 are the factors of 6. What about if we have algebra instead of numbers? Expressions like $x^2 + 3x + 2$ also have factors. They are $(x + 1)$ and $(x + 2)$.

Now that we know what factors are, what exactly does factorising mean? Factorising is the process of finding factors - finding what multiplies together in order to get an expression. Factorising simply "splits an expression" into a multiplication of simpler expressions. It is the direct opposite of expanding i.e. you need to turn an expression into brackets that are multiplied together. For example, splitting the expression $x^2 + 3x + 2$ into $(x + 1)(x + 2)$. We know this is true since if we expand $(x + 1)(x + 2)$ we end up with $x^2 + 3x + 2$.




It's not that easy to find these factors on our own though! It is a bit like baking a cake. Think of expanding as putting all the ingredients together to make a cake and factorising as having the cake already made and having to figure which ingredients were used. It's much easier to build the cake when given the ingredients, right?

Don't worry though, there are techniques to do this and you will see how to find these factors for yourself in chapter 2. You should not be scared of factorising because if you expand your answer and it's the same as the original question, you know whether or not you've factorised correctly! For example, if we expand our answer above of $(x + 1)(x + 2)$ we get back the original question that we were trying to factorise, $x^2 + 3x + 2$.

Factorising is an important way of simplifying algebraic fractions, solving quadratics (it finds the zeros/roots of the equation) and graphing quadratics, so make sure you get good at it! It really is just as simple as learning the rules for each and going through a process of elimination to know which type you have. It will appear over and over again in your course and in life!

# 2 Four Factorisation Types


## 2.1 Type 1: Common Factors

First let's make sure you know what the highest common factor (HCF) is.
Factors are numbers that we can multiply together to get another number. A number can have several factors. For example,

18 has the factors 1,2,3,6,9,18
12 has the factors 1,2,3,4,6,12

To make sure that you never miss any you can always write them out in pairs.
For 18 we can write 1 and 18, 2 and 9, 3 and 6 which then gives us 1,2,3,6,9,18.
For 12 we can write 1 and 12, 2 and 6, 3 and 4 which then gives us 1,2,3,4,6,12.
The **common factors** are those that are found in both lists.
1,2,3,6,9,1
1,2,3,4,6,12
Therefore, the common factors of 12 and 18 are 1,2,3 and 6. The highest common factor is the **greatest** of the common factors which is 6.

This factorising type is recognisable by there being a common factor to EVERY term. We take out the highest common factor aka greatest common divisor.

- For the numbers this is just the highest common factor that you're already familiar with as mentioned above
- For the algebra terms this is the LOWEST power of every term that is common

### 2.1.1 Step By Step Method

Let's go through each of the steps with the example $6x^2 - 4x$

**Step 1**: Find the HCF's of each of the terms. Put this outside of a bracket.

The HCF of the number 6 and 4 is 2
The HCF of the algebra terms $x$ and $x^2$ is $x$
$2x(\quad)$

**Step 2:** Ask yourself what we need to multiply the term outside the bracket by to get each of the terms inside the bracket.

$2x(?+?)$
We need to multiply the above out and end up with $6x^2 - 4x$
$2x$ multiplied by $x$ gives $6x^2$ so we can fill in the first place: $2x(x+?)$
$2x$ multiplied by $-2$ gives $-4x$ so we can fill in the second place: $2x(x-2)$

Our answer is $2x(x-2)$

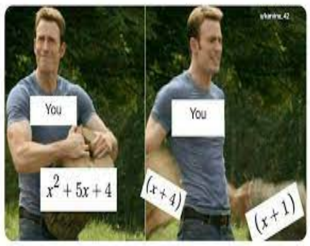
Factorise $4x^2y^2 + 16x^3yz$

Let's colour code the common elements to make this easier to understand: $4x^2y^2 + 16x^3yz$

**Step 1**:
Find the HCF's of each of the colour coded common elements.
The HCF of 4 and 16 is 4. We must pick the highest number, not just any number that fits!
the HCF of $x^2$ and $x^3$ is $x^2$ since 2 is the lowest power
The HCF of $y^2$ and $y$ is $y$ since 1 is the highest power
We can't take out a $z$ term since this is not common to both.
So, in total we take out a 4, an $x^2$ and a $y$
$4x^2y(\quad)$

**Step 2:**
Now we need to ask ourselves what we need to multiply $4x^2y$ by to get both terms
$4x^2y(?+?)$
Hint: we always take out **the lowest power** of the variable common to both terms.
Are you stuck what powers of the variables to use in the bracket? Remember you add the powers when you multiply, so ask yourself what power you need to add you what you have to get what you want
$4x^2y(y+4xz)$
$4x^2y(y+4xz)$

## 2.2 Type 2: Product Sum




This is recognisable by the general quadratic form $ax^2 + bx + c$ where $a = 1$.
**You can see this is not type 1 above since there is no factor common to all terms.**
Let's consider

$$(x - 2)(x + 3)$$

If we expand this we get

$$x^2 - 2x + 3x - 6 = x^2 + 1x - 6.$$

Notice how the '$-6$' was foud by multiplying $-2$ and $3$ and the '$1$' (from $1\ x$) was found by adding $-2 + 3 = 1$. Therefore, if we want to move in the other direction, we must find two numbers that multiply to get $-6$ which in our general form is $c$, and sum to get $1$ which in our general form is $b$. Hence this is called product sum.

### 2.2.1 Step By Step Method

How can a quadratic of the form $x^2 + bx + c$ can be factorised?

Let's go through each of the steps with our previous example $x^2 + x - 6$

**Step 1:** Start out as writing $(x \quad)(x \quad)$

**Step 2:** Find two integers whose product is $c$ and whose sum is $b$.
To do this, we list out all of the possible factors of 'c' and then systematically go through pairs of these until we strike lucky and find a pair that sum to make '$b$'.
$c = -6$
$b = 1$
What two numbers multiply to make $-6$ that add to make $1$?
In other word, we list all the factors of $-6$ and then see which pair adds to give us $1$
$-1, 6$ . The sum to make $5$. Incorrect!
$1, -6$ . These sum to make $-5$. Incorrect!
$-2, 3$ . These sum to make $1$. Correct!!

Note: experience of doing these will make you quicker at spotting the numbers without having to try all the combinations!

**Step 3:** Let's call these two integers $p$ and $q$. One factor is $(x + p)$ and the other factor is $(x + q)$ so we get $(x + p)(x + q)$.
We have $-2$ and $3$
Our factors are $(x - 2)(x + 3)$

Factorise $x^2 + 2x - 8$

Let's colour code to make this easier to understand: $x^2 + 2x - 8$
**Step 1:**
Start by writing $(x \quad)(x \quad)$

**Step 2:**
Now ask what **multiplies** to make $-8$ that **adds** to make $+2$
$+4$ and $-2$
Note: check the numbers that multiply first and then add them after to see whether they are indeed the correct pair

**Step 3:**
Put each of these numbers in a bracket $(x + 4)(x - 2)$
Important: Notice how we want to find the number that make the product first before checking that they are the correct sum.

## 2.3 Type 3: AC Method

This is recognisable by the general form $ax^2 + bx + c$ where $a$ is **no longer equal** to 1.

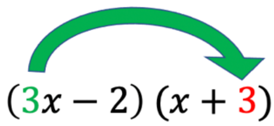
Let's look at an example to get some intuition.

$$(3x - 2)(x + 3)$$

If we expand this we get

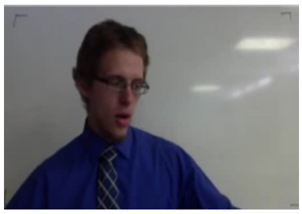
$$3x^2 + 9x - 2x - 6 = 3x^2 + 7x - 6.$$

Notice how the '$-6$' was found by multiplying $-2$ and $3$, BUT the '7' (from $7x$) was NOT found by adding $-2+3$ UNLIKE for type 2 above. This is because of the fact that $3x$ is no longer $x$ like it was for type 2 above where we just had $x$ in bracket $(x \quad)(x \quad)$:
The 3 affects the other 3 and we get a $9x$ instead.




This makes picking the numbers harder than it was in type 2. One option is to do it using trial and error.

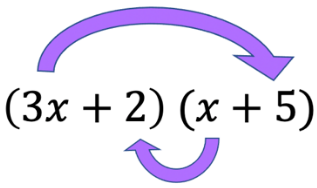
Factoring Using AC Method



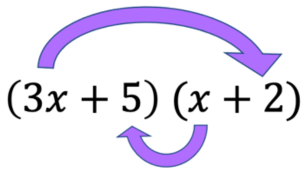
Let's look at the example $3x^2 + 11x + 10$
Our options for our first terms are $(3x \quad)(x \quad)$ or $(-3x \quad)(-x \quad)$
We know need a product of 10 so we can have $-2$ and $-5$, 2 and 5, 1 and 10 and $-1$ and 10.
We also know our middle term $11x$ is a positive so we need numbers that add to a positive number which means 2 and 5 or 1 and 10 are our only options.
Let' try $(3x \quad)(x \quad)$ with 2 and 5

$$(3x + 2)(x + 5)$$

Let's just look at out $x$ terms to see if we get $11x$. We know the other parts will be correct, so it's only the simplification of the 2 middle $x$ terms that we need to worry about.




$15x + 2x = 17x$. We want $11x$, so this is not correct. Let's swap the positions of the 2 and 5.




This works! So, our factorisation is $(3x + 5)(x + 2)$.

This can sometimes take more guesses though. Let's look at another example $6x^2 - 2x - 4$
This could be $(2x \quad)(3x \quad)$ or $(6x \quad)(x \quad)$
We know the product of the two numbers that we chosen must be $-4$ so we can either have $-1$ and 4, 1 and $-4$ or $-2$ and 2.
Let's take some guesses

- $(3x - 4)(2x + 1)$:
  $(3x - 4)(2x + 1) = 6x^2 + 3x - 8x - 4 = 6x^2 - 5x - 4$ WRONG
- $(3x + 4)(2x - 1)$:
  $(3x - 4)(2x + 1) = 6x^2 - 3x + 8x - 4 = 6x^2 + 5x - 4$ WRONG
  Notice how swapping the signs of the numbers from the first guess above gave the same middle number, but the opposite sign. In hindsight, we shouldn't have bothered to try this one.

- $(3x-1)(2x+4)$:
  $(3x-1)(2x+4) = 6x^2 + 12x - 2x - 4 = 6x^2 + 10x - 4$ WRONG
- $(3x+2)(2x-2)$:
  $(3x+2)(2x-2) = 6x^2 - 6x + 4x - 4 = 6x^2 - 2x - 4$ FINALLY!!!!

As you can see this could take a while sometimes and a lot of students struggle to do this under the timed pressure of exams. Luckily there a systematic way that always works!

### 2.3.1 Step By Step Method

Consider the general form for $ax^2 + bx + c$

Let's go through each of the steps with our previous example $6x^2 - 2x - 4$

**Step 1:** Find two integers whose product is $ac$ and whose sum is $b$.
Basically, you multiply the first term by the last term to find $'ac'$ and then list out all of the possible factors of 'ac' and then systematically go through pairs of these until we strike lucky to find a pair that add to make $'b'$.
$ac = 6(-4) = -24$
$b = -2$
What two numbers multiply to make $-24$ that adds to make $-2$?
In other words, list all the factors of $-24$ and then see which pair sum to give us $-2$
$-1, 24$. These sum to 23. Incorrect!
$1, -24$. These sum to -23. Incorrect!
$2, -18$ . These sum to $-16$. Incorrect!
$-2, 18$ . These sum to 16. Incorrect!
$-6$ and 4. These sum to $-2$. Correct!!

Note: experience of doing these will make you quicker at spotting the numbers without having to try all the combinations!

**Step 2:** Let's call the two numbers r and s. We rewrite the middle term as those number: $ax^2 + rx + sx + c$
In other words, the middle term gets split into the two numbers found.
We have $-6$ and 4
We re-write $-2x$ as $-6x + 4x$
$6x^2 - 6x + 4x - 4$

Note: we could also have written $6x^2 + 4x - 6x - 4$ . The order doesn't matter. The final answer will still end up the same in step 5.

**Step 4:** Use 'grouping by pairs' to factor. We take the HCF out of the first two terms and out of the second two terms. In other words we factor the first two terms and the last two terms separately using type 1 factorising.
Let's colour code so which we can see which pairs we consider
$6x^2 - 6x \quad + 4x - 4$
$6x(x-1) + 4(x-1)$

**Step 5:** You can tell you've done it correctly if both brackets end up the same. Meaning, our two new terms have a clear visible common factor .
$6x(x-1) + 4(x-1)$
$x - 1$ is common to both terms so we know we're correct
$(x-1)(6x+4)$

Factorise $3x^2 + 11x + 10$

Let's colour code to make this easier to understand: $3x^2 + 11x + 10$

**Step 1:**
Multiply the first and last numbers together
$3 \times 10 = 30$
What multiples to make 30 that adds to make 11
5 and 6

**Step 2**:
We split the middle term 11 into these 2 numbers found which are 5 and 6
So, our original question $3x^2 + 11x + 10$becomes $3x^2 + 5x + 6x + 10$

Note: It doesn't matter which order we write the numbers. We could have written our middle terms are $6x + 5x$. Our final answer in step 5 will end up the same

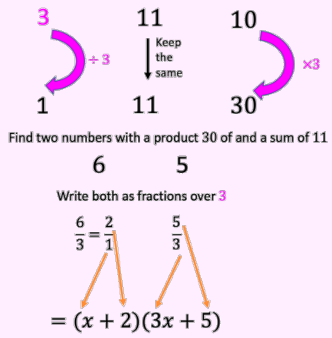
**Step 3:**
Split the terms down the middle in half
$3x^2 + 5x \quad + 6x + 10$

**Step 4:**
Take out what is common **from each side** (in row 1 above)
$x(3x + 5) + 2(3x + 5)$

Note: If there is nothing to take out, just take out a 1. If you have done it correctly the 2 brackets should be the same

**Step 5:** Take out a $(3x + 5)$ common to both
In other words, choose one of the brackets that are the same and the terms in front go in another bracket
$(3x + 5)(x + 2)$

Note: we could also have done this via another method known as the "funnel method"
It is easiest to first see this in action, rather than the list steps.




A step-by-step method would look like:
Step 1: Write the coefficients of the quadratic in a line and divide the first term by $a$ and multiply the last term by $a$
Step 2: Ask yourself what multiplies to make the last number that adds to make the middle number
Step 3: Divide both numbers by $a$
Step 4: Put these numbers into brackets using the pattern shown (numerator is the coefficient of $x$ and the denominator is the constant)

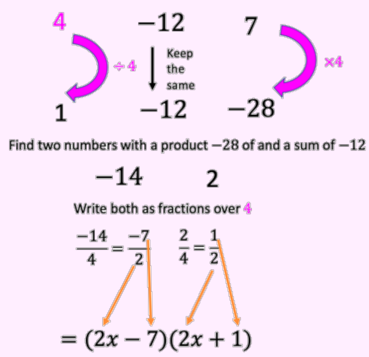
This method requires lot of practice to remember it as it's not natural. Let's take a look at a few more examples

Factorise $4x^2 - 12x - 7$

Let's colour code to make this easier to understand: $4x^2 - 12x - 7$

Multiply the first and last numbers together
$4 \times -7 = -28$
What multiples to make $-28$ that adds to make $-12$ ?
$+2$ and $-14$
We split the middle term $-12$ into the 2 numbers found above, $+2$ and $-14$
So, $4x^2 - 12x - 7$ becomes $4x^2 + 2x - 14x - 7$
Split the terms down the middle in half
$4x^2 + 2x \qquad -14x - 7$
Take out what is common from each side
$2x(2x + 1) - 7(2x + 1)$
Note: Notice how $-7$ was taken out of the second pair since we wanted the bracket to both be $2x + 1$. Taking out a 7 would have given us a bracket of $(-2x - 1)$ which wouldn't have been the same as the first one.
Take out a $(2x - 7)$ common to both
$(2x - 7)(2x + 1)$

Let's see this with the funnel method again




Let's try a final example using the AC method without writing all the steps

Factorise $2x^2 + 7x + 3$

$2 \times 3 = 6$
$2x^2 + 6x \qquad + x + 3$
Take out what is common from each side
$2x(x + 3) + 1(x + 3)$
Note: If there is nothing to take out, we just just take out a 1
$(2x + 1)(x + 3)$

## 2.3.2 Harder Versions of Product Sum and AC Method

Product sum and AC Method won't always be an easy to recognise quadratic. It could be either

- A "hidden quadratic" requiring a substitution. Your hint to recognise this is one power is double the other power.
- Product sum or AC Method but with two variables (for example $x$ and $y$ instead of just $x$)

**Hidden Quadratic Example:**

Factorise $x + 2\sqrt{x} - 3$

Way 1:
$x + 2\sqrt{x} - 3$
$x^1 + 2x^{\frac{1}{2}} - 3$

Use indices rule $(x^a)^b = x^{ab}$ backwards i.e. $x^{ab} = (x^a)^b$

to write $x^1$ as $\left(x^{\frac{1}{2}}\right)^2$

$\left(x^{\frac{1}{2}}\right)^2 + 2x^{\frac{1}{2}} - 3$

$\left(\sqrt{x}\right)^2 + 2\sqrt{x} - 3$

$(\sqrt{x} \quad)(\sqrt{x} \quad)$

$(\sqrt{x} + 3)(\sqrt{x} - 1)$

Way 2:

Instead of using indices rules, it would have been easier to just ask yourself whether the power on one term is double the power of the other and you can easily see this is a hidden quadratic

Use the substitution $y = x^{\frac{1}{2}}$

$y^2 + 2y - 3$

This is now the form you are used to (type 2)

$(y + 3)(y - 1)$

Put your substitution back

$\left(x^{\frac{1}{2}} + 3\right)\left(x^{\frac{1}{2}} - 1\right)$

$(\sqrt{x} + 3)(\sqrt{x} - 1)$

Way 3:

We could have gotten the answer quicker without a substitution

This is type 2 and start with  and proceed as usual

$(\sqrt{x} \quad)(\sqrt{x} \quad)$ , instead of $x$

etc

**Product Sum With 2 Variables Example:**

Factorise $x^2 + 3xy - 4y^2$

$x^2 + 3xy - 4y^2$

Rewrite as $x^2 + 3yx - 4y^2$

$(x \quad)(x \quad)$

What multiples to make $-4y^2$ that adds to make $+3y$

$+4y$ and $-y$

Put each of these in a bracket

$(x + 4y)(x - y)$

**AC Method With 2 Variables Example:**

Factorise $4x^2 + 5xy - 6y^2$

$4x^2 + 5xy - 6y^2$

Rewrite as $4x^2 + 5yx - 6y^2$

We multiply 4 by $-6y^2$ to get $-24y^2$

What multiples to make $-24y^2$ that adds to make $5y$?

$+8y$ and $-3y$

Split the middle term

$4x^2 + 8yx - 3yx - 6y^2$

$4x(x + 2y) - 3y(x + 2y)$

$(4x - 3y)(x + 2y)$

## 2.4 Type 4: Difference Of Two Squares (DOTS)

This type is recognisable by

- Even powers of the variable
- Two terms with a negative sign in the middle. Notice how only type 1 and this type have two terms. Type 1 can have more than two terms though!
- Square rootable numbers

### 2.4.1 Step By Step Method

**Step 1:** Fill into the template

$(\sqrt{1st\ term} - \sqrt{2nd\ term})\ (\sqrt{1st\ term} + \sqrt{2nd\ term})$

**Step 2:** Simplify each $\sqrt{\ }$

Remember square rooting just halves the power!

e.g. $\sqrt{x^6} = (x^6)^{\frac{1}{2}} = x^3$

Factorise $25 - 16x^2$

**Step 1:** $(\sqrt{1st\ term} + \sqrt{2nd\ term})\ (\sqrt{1st\ term} - \sqrt{2nd\ term})$

$(\sqrt{25} + \sqrt{16x^2})\ (\sqrt{25} - \sqrt{16x^2})$

**Step 2:** Simplify each root

$(5 + 4x)(5 - 4x)$

## 2.5 Double Factorisations

You may have to factorise more than once using either of the types above. This is known as double factorisation. Once you've factorised, check each bracket and ask yourself whether you can factorise again?

Factorise $4x^3 - x^2 + 5x$

First take out a common factor

$x(4x^2 - x - 5)$

Next we have AC method

$x(4x - 5)(x + 1)$

Factorise $2n^2 - 50$

First taking out a common factor

$2(n^2 - 25)$

Next factorise by difference of 2 squares

$2(n - 5)(n + 5)$

Factorise $2x^2 + 6x + 4$

First take out a common factor

$2(x^2 + 3x + 2)$

Next we have product sum

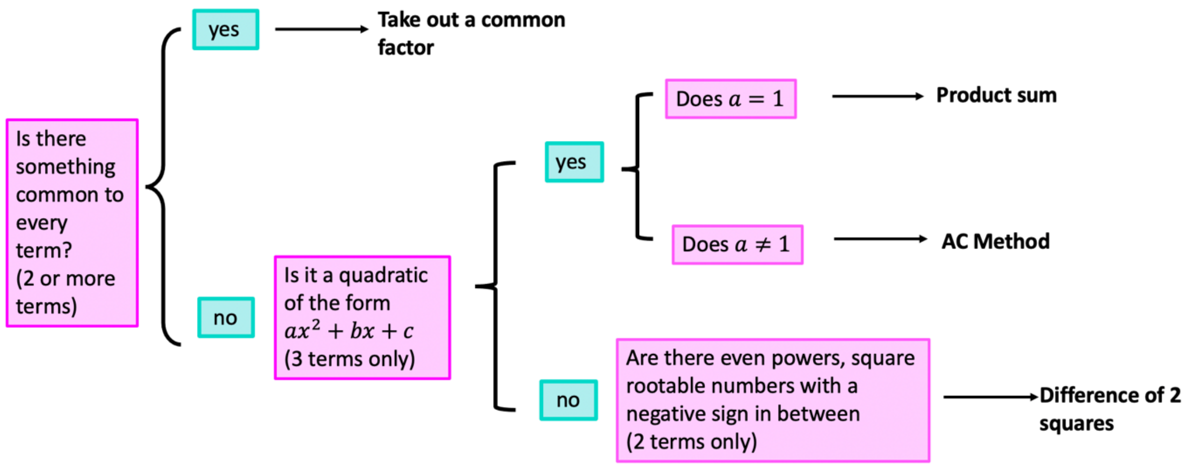
$2(x + 1)(x + 2)$

Factorise $4(x+2)^{\frac{1}{3}} - 5(x+2)^{\frac{4}{3}} + (x+2)^{\frac{7}{3}}$

$4(x+2)^{\frac{1}{3}} - 5(x+2)^{\frac{4}{3}} + (x+2)^{\frac{7}{3}}$
$= (x+2)^{\frac{1}{3}}[4 - 5(x+2) + (x+2)^2]$
$= (x+2)^{\frac{1}{3}}[4 - 5x - 10 + x^2 + 4x + 4]$
Simplify what is inside the square bracket
$= (x+2)^{\frac{1}{3}}[x^2 - x - 2]$
The square bracket is just now type 2
$= (x+2)^{\frac{1}{3}}(x-2)(x+1)$

# 3 Summary Flow Chart

Here is the process of elimination below you should go through in your head AND in this particular order!




**Check whether you can do any of the types again after !**

In other words ask yourself:

1) Can I take out common factors i.e. can I take out something **common** from **every term**?
   **IF NO,**
2) Is it **Product sum?** (with a possible hidden substitution)
   **IF NO,**
3) Is it **AC method**? (with a possible hidden substitution)
   **IF NO,**
4) Is it **difference of two squares**?
   **IF NO,**
   **Give up** ☺

# 4 Calculator Hack

If you're allowed a calculator you can check with factorising types 2-4

Use your equation solver to get the solutions as if the equation were to equal zero

For example, lets say you wanted to factorise $3x^2 - 10x + 8$

Set it equal to $0$

$3x^2 - 10x + 8 = 0$

Use your calculator equation solver to get the solution

Let's say your calculator gave you $x = 2, x = \frac{4}{3}$

Re-arrange each to get $= 0$ and those are your factors

$x = 2$
$x - 2 = 0$

$x = \frac{4}{3}$
$3x = 4$
$3x - 4 = 0$

The factorisation is $(x - 2)(3x - 4)$